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# Understanding Fractions in the Sixth Grade

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# UNDERSTANDING FRACTIONS IN THE SIXTH GRADE

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# INTRODUCTION

During five years of teaching experience at the sixth grade level, I have found that many students lack an understanding of the meaning of fractions. This lack of understanding causes many errors when pupils add, subtract, multiply, or divide fractions. Since some of the difficulty in understanding may be a result of poor teaching, this study is being made to find ways of making fractions meaningful to sixth grade students.

The major effort in this paper will be to develop mathematical meanings and understanding of fractions. Although both the social and mathematical aspects of the teaching of mathematics are important, the writer is limiting himself to a treatment of mathematical phases only. This is not an attempt to write a textbook or to compile a long list of social situations and activities in which fractions occur. Social situations are very important both in the early developmental stages and in the latter stages, and many situations from life should be used to challenge the pupil's competence and to provide for transfer values. Also these social situations can help the pupil respect mathematics for its cultural values as well as for its practical problem solving aspects. The alert teacher should use life situations to accompany the mathematical concepts, principles, and operational procedures.

In a unit on fractions there seems to be several crucial points that many students fail to comprehend. Many times a lack of understanding at these crucial points affects the student's success later on in the unit because of the sequential nature of mathematics. That is, a student's grasp of one concept determines to a large degree his grasp of other related concepts which follow.

In this paper some of the crucial points will be considered. The methods and procedures illustrated in teaching these points are not intended to be exhaustive. They are merely representative of some of the activities and explanations which can be used to develop meanings and understandings. No major attempt is made to give the order of presentation in a unit. The material given should be used by the teacher when the need arises. This need may occur in the original presentation or when reviewing a concept not understood. For instance, if a student has forgotten how to reduce an answer to lowest terms after he has multiplied fractions, the teacher may re-teach the reduction concept by utilizing materials and procedures similar to those employed when this concept was originally presented.

Although the writer has read many books dealing with methods of teaching arithmetic and has examined many textbooks in arithmetic, he has not attempted to give credit through footnotes for ideas and methods outlined in this paper. To have made such an attempt would have resulted oftentimes in a complex maze of names and references at the bottom of a page. Many specialists in arithmetic have been stressing for years some of the ideas suggested in this paper. Furthermore, my adviser, Dr. David J. Davis, suggested many of the ideas. Thus, the writer's role has been to combine ideas from many sources with ideas gleaned from his own experience to make a study and to write a paper that would help him become a better teacher of arithmetic.

The reader should refer to the Bibliography for the list of sources from which the writer has obtained ideas and suggestions.

# CHAPTER I

## ACTIVITIES TO DEVELOP AN UNDERSTANDING OF FRACTIONS

### I. A DESCRIPTION OF SOME OF THE ACTIVITIES

Many different activities can be used to help students understand the meaning of fractions. Chapter I presents a sampling of the many activities the teacher can employ to develop concepts basic to the understanding of fractions. The experiences should progress from concrete to semi-concrete to abstract activities.

The use of concrete objects divided into equal parts. Many different concrete objects, such as apples, pies, oranges, candy bars, ribbons, paper plates, et cetera, can be divided into a number of equal parts. For example, attention can be called to the fact that 1 part of an apple is 1/2 of an apple only if it is 1 of the 2 equal parts into which the whole apple has been divided. Pupils can compare, measure, and manipulate concrete objects until they are able to get 2 equal parts from a whole thing. One of the 2 equal parts then is 1/2 of the whole. The process can be continued in a similar fashion using thirds, fourths, et cetera.

The use of measuring cups and of quart and pint bottles. The concept of a fractional part of a whole may be developed by the use of colored water in measuring cups or in quart and pint bottles. Two 1/2 cups of water can be poured together to make 1 cup. In the same manner three 1/3 cups can be poured together. Two 1/3 cups can be combined to make 2/3 cup. One quart bottle can be filled by pouring 2 pint bottles into it. Thus, 1 pint is 1/2 quart. By similar manipulation the student can understand by first-hand experience what a fractional part of a whole thing really means.

<u>The use of a ruler.</u> A foot ruler marked off into fractional parts of an inch may be used to develop the meaning of a fraction and of equivalent fractions. The diagram of the ruler shown in Figure 1 is marked off in inches, half-inches, fourths, eighths, and sixteenths of an inch. From a foot ruler marked like this pupils find answers to questions

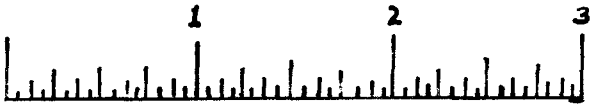
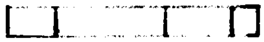



Figure 1

and statements such as the following: (1) How many 1/2 inches are there in an inch? (2) How many 1/4 inches are in an inch? (3) How many 1/4 inches are in 1/2 an inch? (4) Find 4/8 of an inch. (5) How many eighths are in 1/4? (6) Ten-sixteenths is equal to how many eighths? (7) Name two other fractional parts of an inch that are equal to 4/4 of an inch. (8) Find as many different fractions which are equal to each other as you can; for example, 1/2 = 2/4 = 4/8 = 8/16.

<u>The use of a cardboard and rubber bands.</u> In this activity a cardboard ten inches by eight inches in size is given to each pupil in class. By using colored rubber bands on the cardboard, the pupils can show various fractional parts, and the teacher can work with each pupil, asking questions, calling attention to the various parts of the cardboard and to the size of the parts. Obviously the opportunities for teaching and learning are tremendous here. For example, suppose a teacher, after asking his pupils to show fourths on their cardboards, finds one pupil with a board that looks as follows:



. Here the teacher can ask the pupils if the 4 parts are all equal in size. This provides an opportunity for each pupil to estimate the size of parts and to check his estimation with a ruler. The rubber bands can be adjusted until all 4 parts are the same size. Then and then only is each part 1/4 of the board.

The board can also be used to show that 1/2 is larger than 1/3; 1/3 is larger than 1/4; 1/4 is larger than 1/5, et cetera. Figure 2 is a diagram of the way the boards would look when divided into halves, thirds, fourths, and fifths.

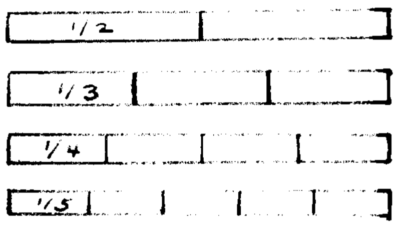



Figure 2

Figure 3 shows that the cardboard can also be used to help the pupil visualize the various shapes the same fractional part of the board can assume. One-half of the board may be


Figure 3

shown by placing the rubber band horizontally or by placing it vertically across the board.

The multiplication of fractions can also be developed meaningfully on these cardboards. One-half of 1/2 can be shown by using rubber bands of 2 colors. The pupil can first show 1/2 of the cardboard by dividing the cardboard into 2 equal parts with a red rubber band placed vertically across the board (see Figure 4). The pupil can then discover the answer to 1/2 of 1/2 by placing a blue rubber band horizontally on the board. The part lettered A in Figure 4 shows that 1/2 of 1/2 is 1/4 of the whole board.

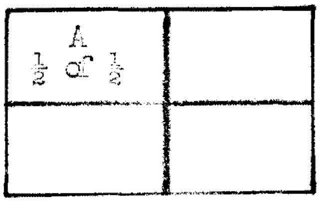



Figure 4

<u>The use of groups of objects.</u> Groups of objects such as pennies, books, toys, marbles, papers, students, and toothpicks can be divided into fractional parts. One

fourth of 8 marbles means that all the marbles are divided into 4 equal groups, with each group having 2 marbles in it (see Figure 5). Thus 1/4 of 8 marbles is 2 marbles. Such

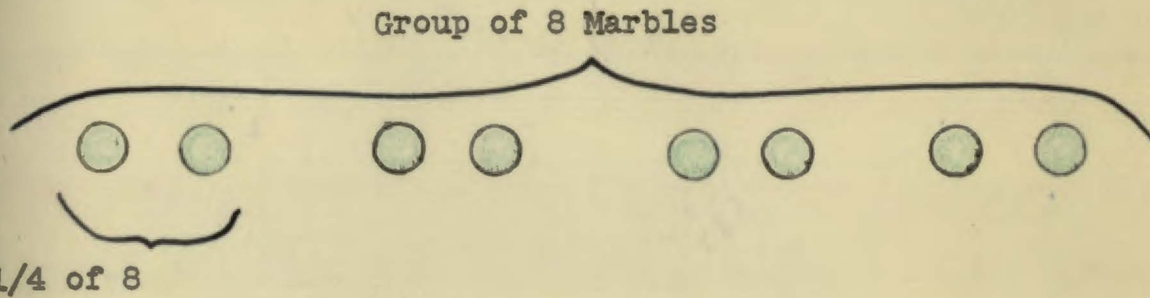



Figure 5

experiences will help the child see that a fraction can be a part of a group of things. The reader will note that the marbles were considered as a single group before the division into 4 equal sub-groups.

<u>The use of class discussions.</u> Class discussions are often valuable means of helping pupils realize how often they use fractions in daily living. The meaning of 1/2-way home, half past 2:00, 1/2 pound, 1/2 dollar, et cetera, might be considered in such discussions. Again, stress is placed on the equality of the size of parts.

<u>The use of diagrams drawn on the blackboard and on paper.</u> After other first-hand experiences with concrete objects similar to those presented earlier in this chapter, the

student can progress to the use of such semi-concrete material as diagrams and pictures; and finally to abstract ideas expressed by the written symbol. Many different diagrams, similar to those presented in Figure 6, can be used to help pupils understand fractions. A different number of parts can

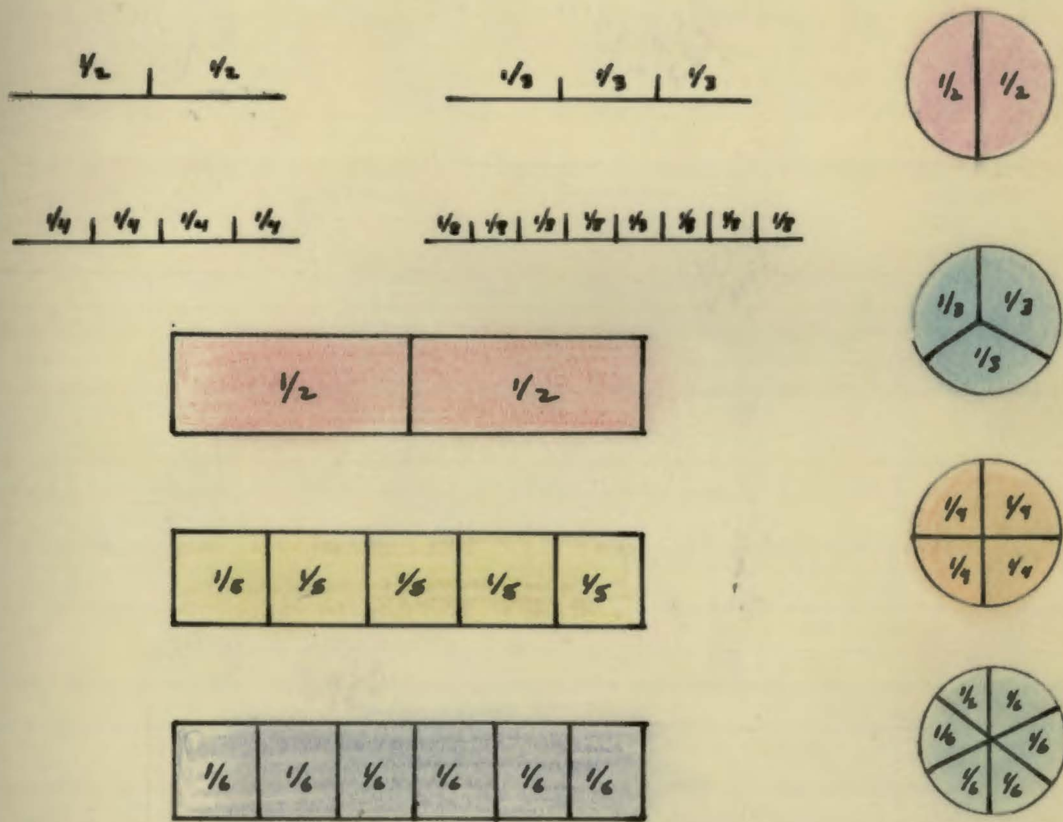



Figure 6

be colored to show the meaning of the numerator. When discussing 1/3 of a circle, the teacher or the students can color one of the 3 equal parts red.

Drawings such as those shown in Figure 7 might be mimeographed on paper or drawn on the blackboard by the teacher. Pupils could be asked to tell or write which figure shows halves, which is divided into thirds or fourths, or which figures do not show fractional parts accurately.

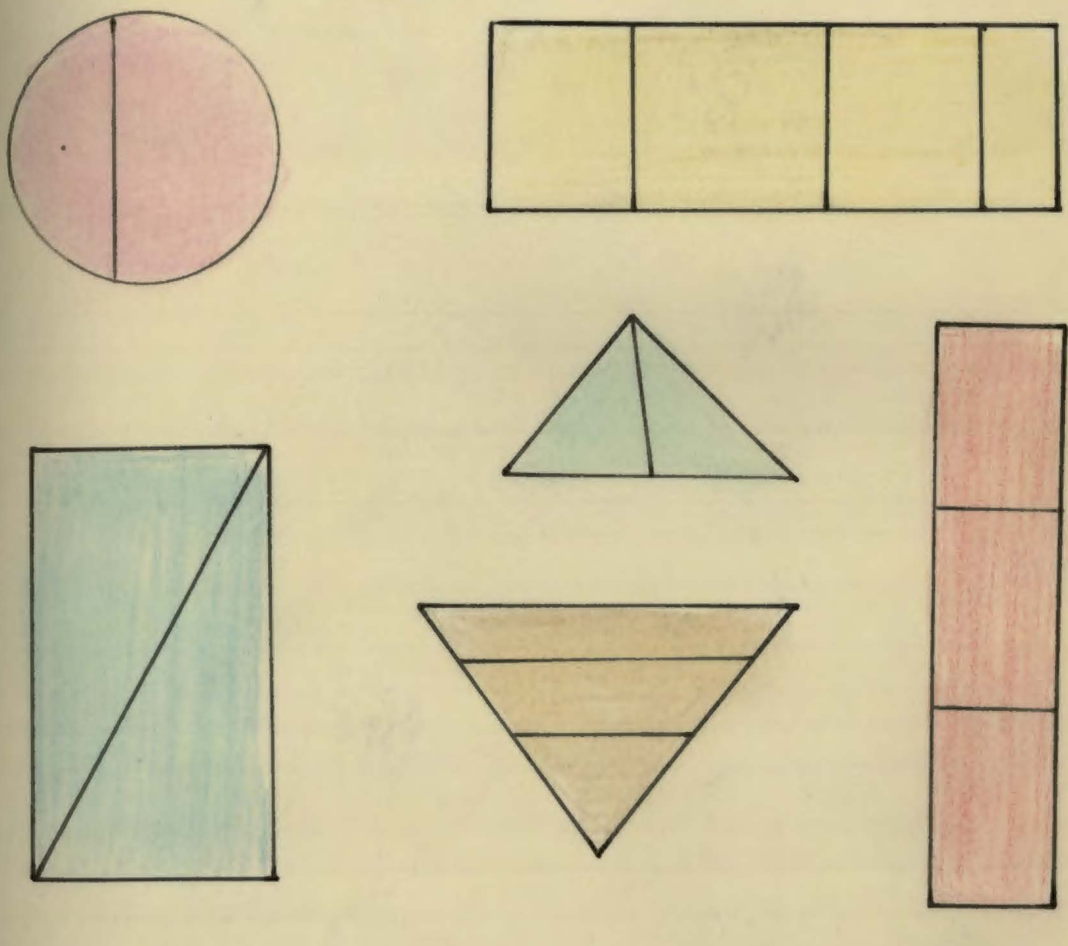
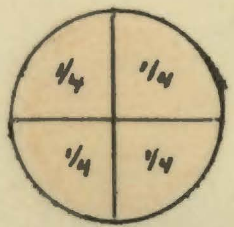


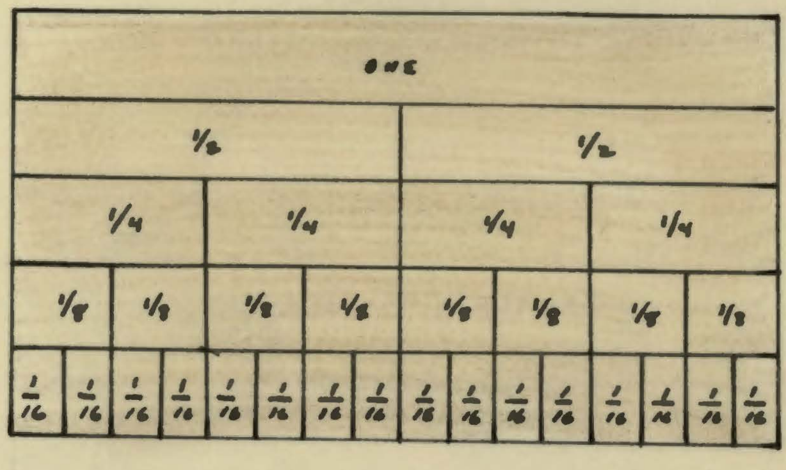
Figure 7

<u>The use of fractional charts and disks.</u> Fractional charts and circular disks are helpful when developing the concept of equivalent fractions. A chart and a disk which

the teacher can construct on large pieces of cardboard for demonstration purposes are shown in Figure 8. The pupils can make similar visual aids on sheets of paper. Other charts and disks can be made to show fractions not included in Figure 8. By using these charts the student can discover fractions which are equal. From a list of equivalent fractions, such




Disk




Chart

Figure 8

as that list presented which follows, the pupils can easily discover that the size of a fraction does not change if the numerator and the denominator are multiplied or divided by the same non-zero number.

1/2 = 2/4 = 4/8 = 8/16

1/4 = 2/8 = 4/16

3/4 = 6/8 = 12/16

1 = 2/2 = 4/4 = 8/8 = 16/16

1/8 = 2/16

3/8 = 6/16

5/8 = 10/16

7/8 = 14/16

1 = 3/3 = 6/6 = 12/12

1/3 = 2/6 = 4/12

2/3 = 4/6 = 8/12

1/6 = 2/12

1/2 = 3/6 = 6/12

5/6 = 10/12

The teacher who uses these or similar activities to make fractions meaningful will have laid a solid foundation of concepts and understandings in terms of which other concepts, skills, and manipulations can be efficiently developed.

## II. BASIC CONCEPTS AND SUMMARY

By participating in activities similar to those presented earlier in Chapter I, the pupils can develop a clear understanding of the basic concepts of fractions. Some of these important concepts will now be considered in greater detail in terms of their meanings and in terms of their relationship to future concepts to be mastered.

<u>Understanding the meaning of the denominator and the numerator.</u> The student's lack of understanding of the roles played by the denominator and the numerator causes many mistakes in fractions. The denominator of a fraction tells the number of equal sized parts into which some whole thing has

been divided, while the numerator tells how many of these equal parts are being used. To gain a clear understanding of the roles played by the numerator and the denominator, the student needs to participate in many of the activities described in the first section of Chapter I. Some of these activities are as follows: (1) Divide concrete objects into equal parts. (2) Pour colored water to determine the size of a cup, a pint and a quart. (3) Use cardboards and rubber bands. (4) Draw diagrams on the blackboard or on paper. (5) Carry on class discussions. These activities are carried out by the students with the teacher's assistance. After many first-hand experiences, the student should be able to discover for himself the meaning of the denominator and the numerator, and the role that each plays in the complete fraction. The importance of these concepts of the numerator and the denominator can hardly be overestimated. Before the students can understand the concepts to be mastered in the addition, subtraction, multiplication and division of fractions, they must understand the meaning of the denominator and the numerator of a fraction.

<u>Understanding the comparative sizes of fractions.</u> The size of each part of a fraction becomes smaller when the number of equal parts into which the whole is divided increases. Thus, if the numerators of two fractions are the same, the fraction with the larger denominator is the smaller. Each fractional part becomes smaller as the number of parts increases. If the

denominators of two fractions are the same, the fraction with the larger numerator is the larger of the two fractions. These two concepts are shown in Figure 9. One-half is larger than 1/4 and 5/6 is larger than 1/6.

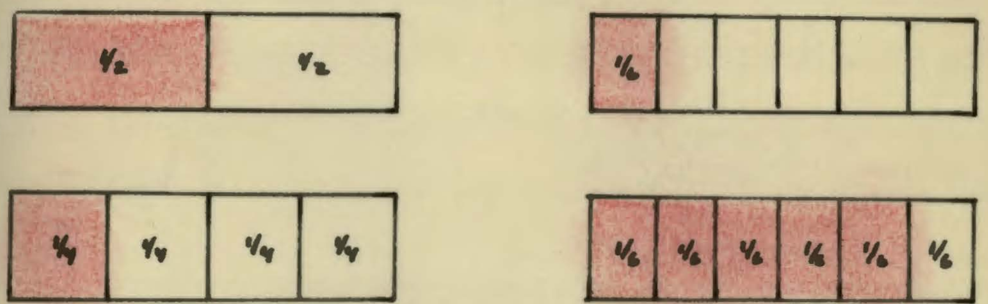


Figure 9

A sound understanding of such concepts as these cannot be achieved by memorization of textbook rules or of rules given by the teacher. The student needs to have many firsthand experiences similar to those described in the first section of this chapter. From the experiences gained by participating in these activities the learner should be able to acquire an understanding of these concepts for himself. Activities involving fractions should not be given too rapidly for the student must have time to make discoveries for himself.

The size of fractional parts should be stressed because the equality and the inequality of parts are of vital importance in the addition and subtraction of fractions. Of first importance here is the necessity for denominators to be of the same size or measure in order to be added. Quarts must be added to quarts, and pints to pints, not quarts to pints. In the same manner it can be shown that 2 quarts and 3 pints cannot be added in their present form. Neither can 2 fourths and 3 eighths be added as these quantities now stand. In both examples the sizes of the measures or parts are different. Before adding 2 quarts and 3 pints the students must change 2 quarts to 4 pints. Then the 2 quantities can be added because both involve the same measure; namely, pints. In similar fashion 2 fourths and 3 eighths cannot be added until 2 fourths is changed to 4 eighths. The two fractions then both involve eighths, which are parts of the same size. Thus 4 pints plus 3 pints equals 7 pints, just as 4 eighths plus 3 eighths equals 7 eighths.

The mathematical way of writing 4 eighths plus 3 eighths equals 7 eighths is $4/8 + 3/8 = 7/8$. It should be noted that the denominators of the fractions are not added anymore than pints and pints are added. The denominator tells us the size of each part that is to be added just as pints tells us the size of each part. It is necessary to have the size of each part the same before adding. Thus, the denominators of two fractions must be the same. When this occurs, the students are told that the fractions have what is often called a "common denominator." To illustrate further that the size of the units being added must be the same, the teacher can have a student add 1 yard and 1 foot by drawing a line on the blackboard that is 1 yard and 1 foot in length. The class can then see that the line is neither two yards nor two feet long. Other units such as a nickel and a dime, an hour and a minute, or a quart and a gallon can be used to illustrate the importance of the sizes of quantities and parts that are to be added.

Thus it can be shown that a common denominator is needed before fractions can be added or subtracted because the size of the units to be added or subtracted must be the same. Many times this concept is withheld from the students by teachers who simply say that 1/2 and 1/4 cannot be added as they stand because that would be like adding apples and oranges. The teacher is then at a loss to explain who 1-half apple cannot be added to 1-fourth apple in the present form of these fractions. Here both fractions involve apples, and they still cannot be added as they are. The pupils must understand that before they can add 1-half apple and 1-fourth apple, it is necessary to express each fraction in terms of parts of the same size. Thus, we can express each fraction in terms of fourths of an apple and we then are combining 2-fourths apple with 1-fourth apple, each fraction involving parts of the same size; namely, fourths of an apple. The teacher who tells pupils they must get a common denominator before they can add or subtract fractions, without teaching "why they must" is failing to teach meaningfully. When fractions have a common denominator, they have parts of the same size, and the equality of the parts involved is the crucial consideration.

Understanding equivalent fractions. Equivalent fractions are fractions which are equal but have different numerators and different denominators. One-half is equal to 2/4 but the numerators and the denominators are different.

The students need to participate in activities which will help them understand that 1/2 = 2/4. Concrete objects of the same size can be divided into halves and into fourths so the student can compare the sizes of 1/2 and 2/4. Rubber bands on two cardboards the same size will illustrate the same concept. A fractional chart such as is shown in Figure 8, page 11, can be used to find that 1/2 is equal to 2/4.

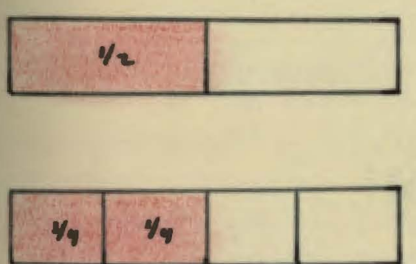



Figure 10

The diagram in Figure 10 illustrates another way of showing that 1/2 = 2/4. By participating in the activities just mentioned, the student will be able to discover that 1/3 = 2/6, 1/2 = 5/10, 3/4 = 12/16, and so on.

The student should be able to discover from a list of equivalent fractions that the size of a fraction does not change when both the numerator and the denominator are multiplied or divided by the same non-zero number. For instance,

$1/3 = \frac{2}{2} \frac{x}{x} \frac{1}{3} = 2/6$. Also $2/6 = \frac{2}{6} \frac{\div}{\div} \frac{2}{2} = 1/3$. However, it is very important that the student does not learn to perform these two processes by rule only. He must understand them through concrete and semi-concrete experiences with fractions.

The concept of equivalent fractions is used and developed meaningfully throughout the entire unit on fractions. Answers to addition, subtraction, multiplication, and division problems can often be expressed in more simple form by dividing the numerator and the denominator by the same non-zero number. When fractions are expressed with a common denominator, the numerators and the denominators of the fractions are multiplied by the same non-zero number.

Understanding improper fractions. By using rubber bands on cardboards, a ruler marked off into fractional parts, concrete objects which can be divided into equal-sized pieces, and diagrams on the blackboard and on paper, and other such materials, the teacher can show the meaning of a proper fraction and an improper fraction. The student needs to understand that the denominator tells the number of equal sized parts into which the whole has been divided. The numerator tells us how many of these equal sized parts are being used. Any fraction which is less than one whole is a proper fraction. With rubber bands on a cardboard the teacher can show that 2/3 means that the whole board is divided into 3 equal parts and 2 of these equal parts are being used.

Fractions that are equal to or larger than one whole are called improper fractions. The teacher can show 5/3 with the rubber bands on cardboards (see Figure 11). One cardboard

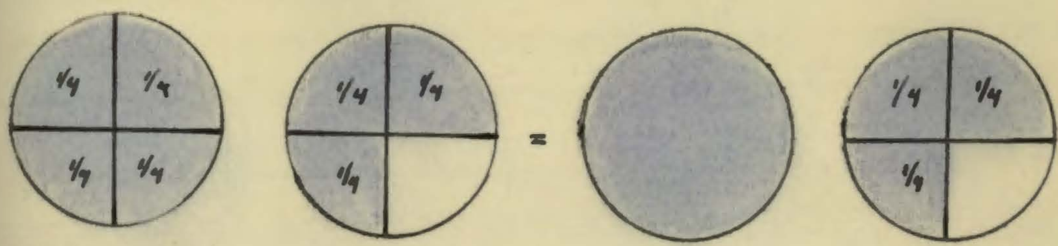

$5/3 = 3/3 + 2/3$

$5/3 = 1 + 2/3 = 1\frac{2}{3}$

Figure 11

has just 3 equal sized parts because the denominator is 3. But there are still two more equal parts to use, so another cardboard is divided into 3 equal parts. Two of the 3 equal parts are used on the second board. By actually manipulating the rubber bands the student should be able to see that 5/3 is equal to 1 whole board and 2/3 of another. Another way to write this is $1\frac{2}{3}$ .

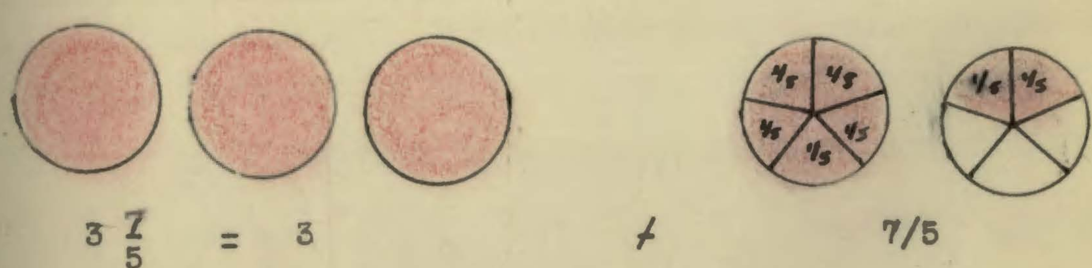
Semi-concrete material in the form of diagrams and pictures may be used to show the concept of an improper fraction. For example, the diagram in Figure 12, on page 20, shows that 7/4 is equal to 4/4 of 1 circle + 3/4 of another circle. Thus $7/4 = 4/4 + 3/4 = 1 + 3/4 = 1\frac{3}{4}$ . Through these activities involving concrete and semi-concrete materials, the students can be led to discover that an improper fraction may be changed to a whole or mixed number by dividing the numerator by the denominator. Seven-fourths $= 7 \div 4 = 1\frac{3}{4}$.




$7/4 = 4/4 \quad + \quad 3/4 \quad = \quad 1 \quad + \quad 3/4 = 1\frac{3}{4}$

Figure 12

Many times after adding fractions the answer will be a mixed number involving an improper fraction. An example is $3\frac{7}{5}$. In order to understand that $3\frac{7}{5} = 4\frac{2}{5}$, the student needs to see that $3\frac{7}{5} = 3 + 7/5$. A review of the place value of the whole numbers may help at this point. Two hundred fifty-four means $200 + 50 + 4$. In similar manner it can be shown that $3\frac{7}{5}$ is equal to $3 + 7/5$. The improper fraction 7/5 can now be changed to $1\frac{2}{5}$. The substitution of $1\frac{2}{5}$ for 7/5 in $3 + 7/5$ results in $3 + 1\frac{2}{5} = 4\frac{2}{5}$. Figure 13 illustrates these ideas.




$3\frac{7}{5} \quad = \quad 3 \quad + \quad 7/5$

$3 \quad + \quad 1\frac{2}{5} \quad = \quad 4\frac{2}{5}$

Figure 13

Changing a mixed number to an improper fraction involves a process just the opposite of that involved in changing an improper fraction to a mixed number. The use of cardboards and rubber bands will help the students develop understanding. The teacher can begin by showing that 7/4 is equal to $1\frac{3}{4}$, (see Figure 14). Two boards can be divided into 4 equal parts.


$$7/4 = 4/4 + 3/4$$

$$7/4 = 1 + 3/4 = 1\frac{3}{4}$$

Figure 14

All 4 parts of one board and 3 of the 4 parts on the second board will have to be used. Thus $1\frac{3}{4}$ boards have been used. Now the students should be able to see from their boards that $1\frac{3}{4} = 4/4 + 3/4 = 7/4$ .

Another means of developing an understanding of how to change a mixed number into an improper fraction is to begin by changing whole objects into fractional parts. Each whole object can be divided into 2/2, 3/3, 4/4, and so on. Three objects would equal 6/2, 9/3, 12/4 (see Figure 15 on page 22).

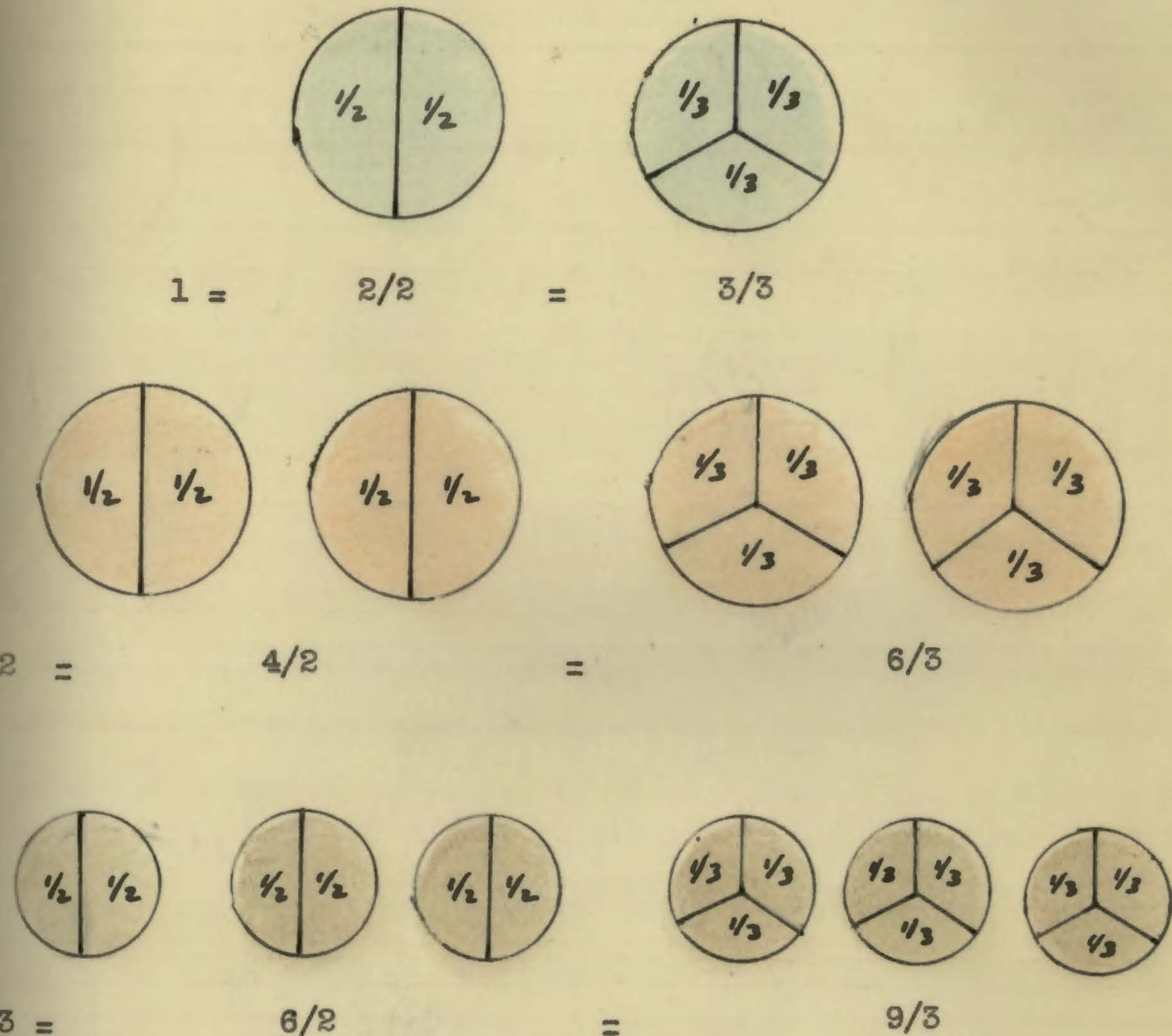
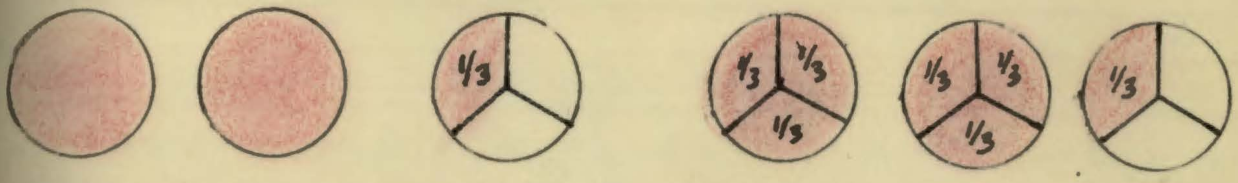



Figure 15

In a similar manner it can be shown that any whole number may be changed into an improper fraction. The numerator of the improper fraction is equal to the whole number times the denominator of the improper fraction. Five = 10/2 = 15/3, and so on. The mixed number $2\frac{1}{3}$ is equal to 2 + 1/3. Since 2 = 6/3,

$2\frac{1}{3} = 6/3 + 1/3 = 7/3$. Figure 16 illustrates this.




$$2\tfrac{1}{3} = 3/3 + 3/3 + 1/3 = 7/3$$

$$2\tfrac{1}{3} = 6/3 + 1/3 = 7/3$$

Figure 16

Chapter I, which is now being concluded, has laid the foundation for a meaningful approach to the operational procedures with fractions. The basic concepts comprising this foundation were developed through pupil participation in many concrete, semi-concrete, and abstract activities. In Chapters II, III, and IV, which are concerned with the fundamental operations of fractions, the writer assumes that the teacher has developed the basic concepts stressed in Chapter I. This assumption is made to avoid needless repetition.

CHAPTER II

UNDERSTANDING THE ADDITION AND SUBTRACTION OF FRACTIONS

In Chapter II, some of the basic concepts necessary to understand the addition and subtraction of fractions are considered. All concepts involved in these fundamental processes are not discussed, merely those basic concepts which are encountered at certain crucial points. The basic concepts at the crucial points must be understood by the students if the fundamental processes are to be meaningful. Chapter II is written on the assumption that the teacher has laid the foundation of basic concepts considered in Chapter I. This assumption is made to avoid discussing repeatedly the activities and the concepts which actually are basic to all operational procedures with fractions.

I. UNDERSTANDING THE ADDITION OF FRACTIONS

Addition of like fractions. Like fractions are fractions with denominators of the same size. For instance, 1/4 and 3/4 are like fractions because both have 4 as the denominator. When the denominators of fractions are the same, the size of each fractional part is the same. Figure 17 shows two circles that are the same size. One-fourth of one circle

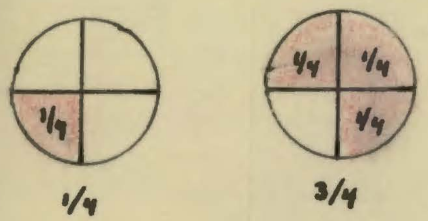


Figure 17

is colored red while the other has 3/4 colored. Both circles are made up of fractional parts that are the same size; namely 1/4's.

One important concept the teacher must develop to make the addition of like fractions meaningful to pupils is that the numerators of the fractions are added and the denominators are not. Before this concept can be used with meaning, the student must understand that the denominator of a fraction tells the size of each of the equal parts into which some whole thing has been divided, while the numerator tells how many of the parts of that size are being used. (See pages 12 and 13 for a more complete treatment of this concept.) Many concrete, semi-concrete, and abstract activities should be used to give meaning to the addition process. Concrete objects such as apples, oranges, paper, wooden blocks, and the like can be divided into fractional parts. Then the sum of the fractional parts of the same size can be found by actually manipulating the fractional parts. For example, $1/4 + 2/4 = 3/4$. The student must be led to understand that the number of equal parts being used are added while the size of each part, which is 1/4, does not change and is not added. Thus, the numerators are added but the denominators are not.

The following exercises are illustrative of the types of exercises which can be employed to make the addition of like fractions meaningful to pupils:

1 quart + 2 quarts = 3 quarts

1 fourth + 2 fourths = 3 fourths

$1/4 + 2/4 = 3/4$

In the classroom activity, the teacher could give the three exercises listed above, but not the answers. The pupils would be asked to get their own answers, which for some pupils might well involve the utilization of concrete objects which the teacher should always have available. The discussion period, which occurs after answers are obtained, should focus attention on the similarity of the three exercises and on the fact that the denominators (4ths) of the fractions are not added any more than "quart" and "quarts" are added. Only the 1 and the 2 are added. The "quart" and the "fourth" tell merely the size of the quantities and are carried along as what might be called labels.

In early phases of teaching the addition of fractions, it is helpful to write the fractions in both the word form and in the mathematical form. For example:

3 eighths + 4 eighths = ?

$3/8 + 4/8 = ?$

Some other exercises similar to those related to quarts are:

2 feet + 3 feet = 5 feet

2 eighths + 3 eighths = 5 eighths

$2/8 + 3/8 = 5/8$

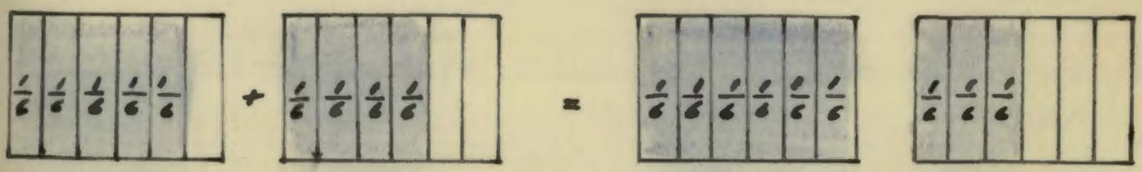
4 ounces + 7 ounces = 11 ounces

4 sixteenths + 7 sixteenths = 11 sixteenths

4/16 + 7/16 = 11/16

The reader will note that in each preceding exercise, quantities or parts of the same size are to be added. Teachers usually characterize such exercises as having a "common denominator." Such a characterization, without stress on the equality of the size of parts which exists when there is a common denominator, can remove meaning from the situation to such a degree that pupils may be handicapped in all later work with fractions. For further discussion of the denominator as a unit of measure or size, see pages 13-16.

Cardboards with rubber bands can be used to show concretely why the numerators but not the denominators of like fractions are added. When adding 1/3 + 1/3, for example, a cardboard may be divided into thirds. The student can then see that 1/3 + 1/3 = 2/3. If 5/6 + 4/6 are added, as in Figure 18, one cardboard can be used to show 5/6 and another board to show 4/6. When the two fractions are combined, the

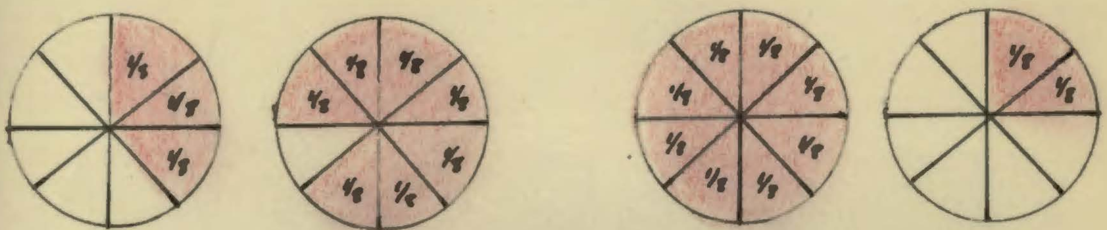


5/6 + 4/6 = 9/6 = $1\frac{3}{6}$ = $1\frac{1}{2}$

Figure 18

students can find by counting that nine of the 1/6's are being used. From the same example it can be seen that 9/6 = 1 whole board + 3/6 of another board, or 1 $\frac{1}{2}$ boards.

Diagrams may be used to picture the addition of like fractions. Figure 19 shows by the use of circles the addition of 3/8 and 7/8. Ten-eighths is equal to 1 whole + 2/8 or 1 $\frac{1}{4}$.



3/8 + 7/8 = 10/8 = 1 $\frac{2}{8}$ = 1 $\frac{1}{4}$

Figure 19

The students should discover by actual participation in many concrete, semi-concrete, and abstract activities similar to those suggested on pages 25-27 that the numerators of like fractions are added but the denominators are not. The teacher should not give this rule to the student as something that is to be blindly memorized.

<u>Adding unlike fractions.</u> The basic concept of comparative sizes of fractions is very important in understanding the addition of unlike fractions. For a discussion of the comparative sizes of fractions, see pages 13-16.

Unlike fractions are fractions with different denominators. This means that the numbers of equal parts into which two wholes are divided are different. One-half means a whole is divided into 2 equal parts and 1/4 means a whole is divided into 4 equal parts. In Figure 20 it can be seen that 1/2 is larger than 1/4.

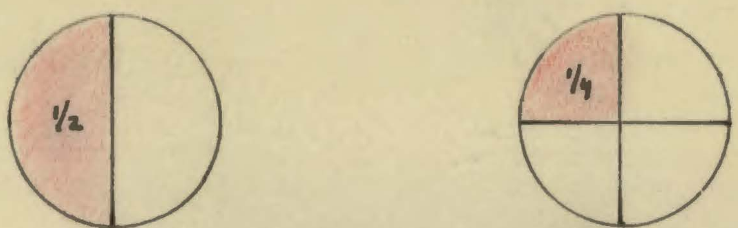



Figure 20

The teacher can now consider the exercise of adding unequal units of measure. For example, 1 foot + 1 yard does not = 2 feet or 2 yards. Neither does 1/2 + 1/4 = 2/2 or 2/4 because each fraction has a different unit of measure. If 1 yard is changed to 3 feet, they can be added to 1 foot to get 4 feet. (1 yd. + 1 ft. = 3 ft. + 1 ft. = 4 ft.) If 1/2 is changed to 2/4, it can be added to 1/4 to get 3/4. (1/2 + 1/4 = 2/4 + 1/4 = 3/4)

Before pupils will understand the concept that only quantities or parts of the same size can be added, the teacher must call attention to the fact that units of length must be the same size before a number of those units can be added. For example, 1 foot + 3 feet can be added to get 4 feet because the unit of measure in both quantities is the same (foot). But 2 feet and 2 inches cannot be added by combining 2 and 2 because the units of measure in these cases are different. To show the student that 2 feet + 2 inches does not equal 4 inches or 4 feet, the teacher can have the students draw on the blackboard a line equal in length to 2 feet + 2 inches. Now the student can actually measure the line to find its length. If 2 feet are changed to 24 inches, the problem becomes 24 inches + 2 inches, which equals 26 inches. Both the 24 and the 2 now involve the same unit of measure (inches), and can be added by combining 24 and 2. See Figure 21.


2 ft. + 2 in. = 24 in. + 2 in. = 26 in.

Figure 21

The concept of changing measures of different sizes to measures of the same size can be applied to the addition of fractions when the denominators are different. When the denominators of two fractions are different, the size of each fractional part of the whole is different (see Figure 20, page 29). The unlike fractions must be changed to like fractions before the fractions can be added. Just as 2 feet must be changed to 24 inches before 2 feet can be added to 2 inches, so 1/2 must be changed to 2/4 before it can be added to 1/4. This relationship may be presented in the following form:

2 feet + 2 inches = 24 inches + 2 inches = 26 inches

1 half + 1 fourth = 2 fourths + 1 fourth = 3 fourths

1/2 + 1/4 = 2/4 + 1/4 = 3/4

Many other exercises similar to those following may be introduced and discussed:

3 quarts + 2 pints = 6 pints + 2 pints = 8 pints

2 pounds + 8 ounces = 32 ounces + 8 ounces = 40 ounces

6 feet + 9 inches = 72 inches + 9 inches = 81 inches

3 dimes + 2 nickels = 6 nickels + 2 nickels = 8 nickels

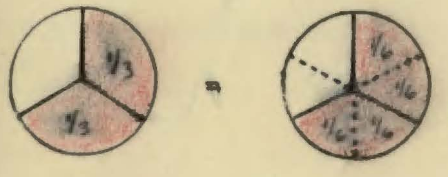
4 gallons + 3 quarts = 16 quarts + 3 quarts = 19 quarts

2 thirds + 1 sixth = 4 sixths + 1 sixth = 5 sixths

2/3 + 1/6 = 4/6 + 1/6 = 5/6

A discussion of the above exercises should bring out the fact that in each exercise the addition could not be performed immediately because the units of measure were of different sizes. Next, the student's attention should be called to the fact that the units of different size are changed to units of the same size. Three quarts are changed to 6 pints. The 6 pints can then be added to 2 pints because both quantities involve the same units of measure, pints. (Six pints + 2 pints = 8 pints.) The same reasoning may be used for other examples. Special attention, however, needs to be given to

2 thirds + 1 sixth. The unit of measure in 2 thirds is thirds but in 1 sixth, the unit of measure is sixths. Thus as different units of measure, they cannot be added as they stand. But as Figure 22 shows, 2 thirds = 4 sixths. Now the problem is 4 sixths + 1 sixth = 5 sixths. If the denominators were written with figures instead of words, 2 thirds + 1 sixth = 4 sixths + 1 sixth = 5 sixths would look like this: 2/3 + 1/6 = 4/6 + 1/6 = 5/6. Other examples similar to the following will help the student to understand that unlike fractions are changed to like fractions before addition can take place:




2/3 = 4/6

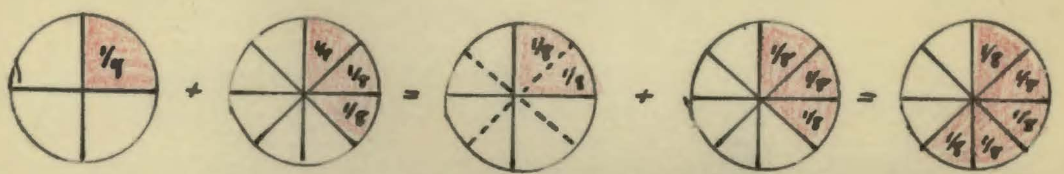
Figure 22

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 fourths | + | 5 eighths | = | 6 eighths | + | 5 eighths | = | 11 eighths |
| 3/4 | + | 5/8 | = | 6/8 | + | 5/8 | = | 11/8 |
| 2 fifths | + | 3 tenths | = | 4 tenths | + | 3 tenths | = | 7 tenths |
| 2/5 | + | 3/10 | = | 4/10 | + | 3/10 | = | 7/10 |

A "common denominator" is the denominator of two or more like fractions. Because fractions must have the same unit of measure before they can be added, the denominators must be the same. In other words, fractions must have a "common denominator" before they can be added. Unlike fractions must first be changed to like fractions before addition can be completed.

A great deal of emphasis must be placed upon <u>why</u> unlike fractions cannot be added as they stand. The student must understand by actually participating in concrete activities that measures of unequal sizes cannot be added until they are changed to measures of the same size.

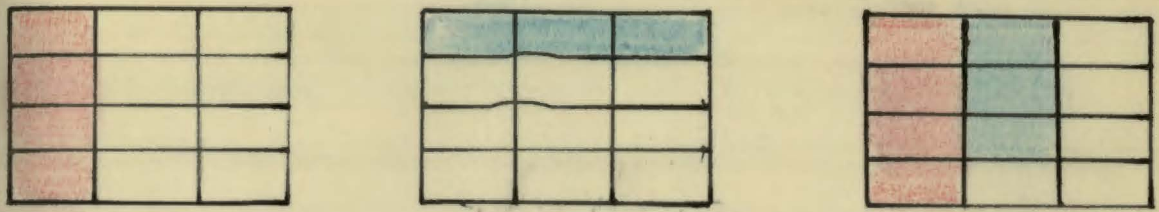
Pictures can be used to develop the understanding necessary to add unlike fractions. Figure 23 shows a picture of



1/4 + 3/8 = 2/8 + 3/8 = 5/8

Figure 23

1/4 + 3/8. The 1/4 is changed to 2/8 and added to 3/8 to get 5/8. In Figure 24, page 34, 1/3 is to be added to 1/4. One-third of a rectangle is colored red and 1/4 of another rectangle of the same size is colored blue. Since 1/3 and 1/4 are unlike fractions, they cannot be added until they are changed to like fractions; that is, to fractions involving units of the same size. Both 1/3 and 1/4 can be changed to twelfths. One third = 4/12 and 1/4 = 3/12. Now 4/12 can be added to 3/12 to get 7/12. Other diagrams similar to those in Figure 23 and Figure 24, can be drawn by the students to show the answers to 2/3 + 1/4; 1/2 + 1/5; 1/3 + 3/8; 3/4 + 5/12, and so on.

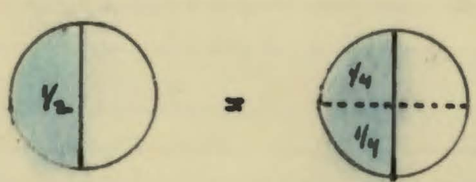


1/3 + 1/4 = ?

4/12 + 3/12 = 7/12

Figure 24

Pages 10-11 and 17-18 present a detailed explanation of activities that can be used to make meaningful the concept of raising the terms of a fraction. "Raising the terms of a fraction" means that the numerator and the denominator of a fraction are increased but the new fraction is not changed in size (1/2 = 2/4). Figure 25 shows that 1/2 and 2/4 are the same size. By participating in such activities the student



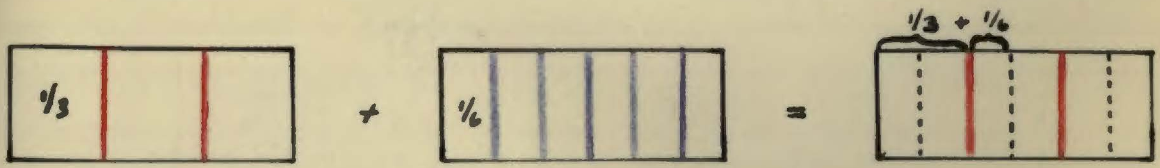

1/2 = 2/4

Figure 25

should discover that the numerator and the denominator of a fraction can be multiplied by the same non-zero number without changing the value of the fraction.

Even after students understand why unlike fractions cannot be added until they are changed to like fractions, and how the terms of a fraction are raised, they may still have difficulty determining a common denominator in certain exercises. The students may recognize that 1/3 + 1/6 cannot be added as they stand because the size of the fractional part in 1/3 is not the same as in 1/6, but they may not understand how to find a common denominator for 1/3 + 1/6 so that 1/3 and 1/6 can be made like fractions. Many activities involving concrete and semi-concrete materials should be used to help the student understand how to determine a common denominator for unlike fractions.

One activity that may be used to help the children determine a common denominator for unlike fractions involves the use of cardboards and colored rubber bands. As shown in Figure 26, two cardboards may be used to show 1/3 and 1/6.

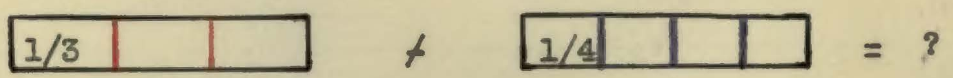



1/3 + 1/6 = 3/6 or 1/2

Figure 26

One board is divided into thirds with red bands and the other is marked into sixths with blue bands. The students can easily see by observing their boards that the 1/3 size piece can be changed to 2 of the 1/6 size pieces by placing a blue rubber band between the red bands. However, 1/6 cannot be changed to thirds because 1/6 is already smaller than 1/3. Now the students can add 2/6 + 1/6 to get 3/6 because both 2/6 and 1/6 are fractional parts of a sixth size. The answer is 3/6 or 1/2.

In the example 1/3 + 1/4, two cardboards may be used to represent 1/3 and 1/4. As shown in Figure 27, one board will have red bands to show thirds and the other will have blue bands to show fourths. As seen from the boards, 1/4 is




1/3 + 1/4 = ?

Figure 27

smaller than 1/3, so 1/4 cannot be changed to thirds. But neither can 1/3 be changed to 1/4. Thus, both 1/3 and 1/4 must be changed to fractions with the same denominator before they can be added but the common denominator is not 3 or 4.

The students need to experiment with their boards divided into thirds and fourths until they find a common denominator for 1/3 and 1/4. The experimentation might begin by placing green rubber bands between the red bands on the board showing thirds. These green bands, as shown in Figure 28, page 37, are placed so that each 1/3 is divided into 2 equal parts. Now the board is divided into 6 equal parts and 1/3 of the board contains 2 of the equal parts. Thus 1/3 = 2/6. But 1/4 cannot be changed into sixths as students can see when green bands are placed on the board showing fourths. Each 1/4, when divided into 2 equal pieces becomes 2/8 instead of 2/6. Thus it can be seen from the boards that the common denominator for 1/3 and 1/4 is not 6 or 8. The experimentation may continue

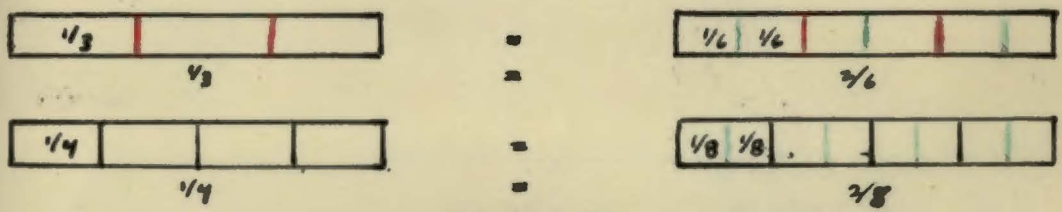



Figure 28

by using the board showing fourths as illustrated in Figure 29. Each 1/4 can be divided into 3 equal parts by placing 2 green rubber bands on each 1/4. As can be seen, the whole board is

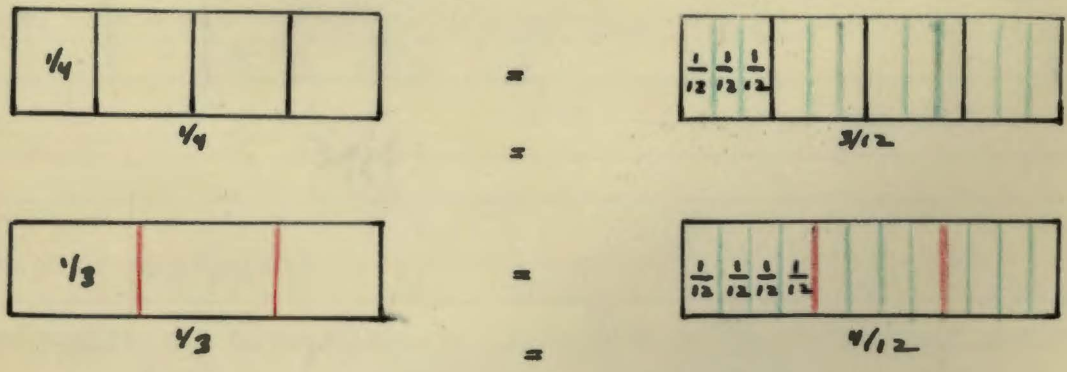
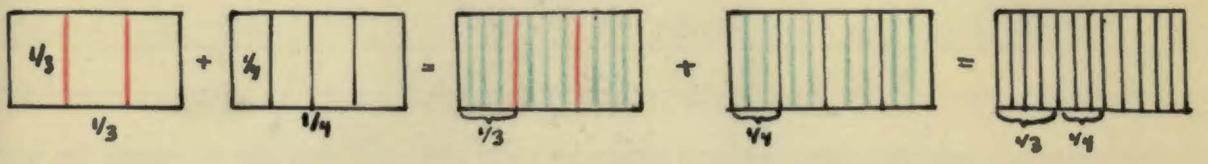



Figure 29

then divided into twelfths and each 1/3 has 3/12 in it. Now green bands are placed on the board showing thirds to see if the 1/3 pieces can be made into 1/12 pieces. As can be seen after experimentation and in Figure 29, 1/3 = 4/12. Thus, 1/3 and 1/4 have a common denominator of 12. As shown in Figure 30, 1/3 ≠ 1/4 = 4/12 ≠ 3/12 = 7/12. The student has discovered by actual experimentation with the boards that 12 is the common denominator of 1/3 and 1/4.




1/3 ≠ 1/4 = 4/12 ≠ 3/12 = 7/12

Figure 30

Another way to show that 12 is a common denominator in the problem 1/3 ≠ 1/4 is to use just one cardboard as shown in Figure 31. Red rubber bands are placed horizontally across the board to divide it into thirds. Blue bands are then placed vertically so that the board is divided into fourths. The student can now count the number of equal sized small parts on the entire board. Since there are 12 equal sized parts, each part is 1/12 of the whole board. Now the student can count the number of twelfths in 1/3 and in 1/4. They will find that 1/3 = 4/12 and 1/4 = 3/12. Thus 12 is the common denominator for 1/3 and 1/4 (1/3 ≠ 1/4 = 4/12 ≠ 3/12 = 7/12).

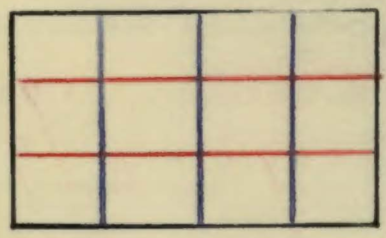


Figure 31

The same procedures used in showing 1/3 ≠ 1/6 = 2/6 ≠ 1/6 = 3/6 or 1/2 and in showing 1/4 ≠ 1/3 = 3/12 ≠ 4/12 = 7/12 can be used to show on cardboards with rubber bands that 3/4 + 5/8 = 6/8 + 5/8 = 11/8 or 1 $\frac{3}{8}$ ; 1/5 + 1/3 = 3/15 + 5/15 = 8/15 ; 1/6 + 1/9 = 3/18 + 2/18 = 5/18 ; 1/8 + 1/6 + 1/4 = 3/24 + 4/24 + 6/24 = 13/24. Many other examples similar to those just listed may be solved by using colored rubber bands on cardboards. The students can experiment with their boards until they find a common denominator for the unlike fractions they are to add.

Another activity that will help the students understand how to find a common denominator for unlike fractions involves the use of concrete objects such as apples, oranges, pieces of paper, etc. One piece of paper can be divided into thirds, and another piece the same size can be divided into sixths. Since 1/3 + 1/6 cannot be added as they stand because each fraction represents a different size unit of measure, they must be changed to like fractions with a common denominator so that both fractions represent units of measure of the same size. It can easily be seen by the student that 1/6 of the piece of paper cannot be changed into 1/3 of the piece of paper because 1/3 is larger than 1/6. But 1/3 of the piece of paper can be made into 2/6 of the paper by cutting the 1/3 piece into 2 equal pieces. Now the students can see by actually manipulating the parts that 2 pieces each of 1/6 size, or 2/6, is equal to 1/3. Thus 1/3 = 2/6. In the problem 1/3 + 1/6, 2/6 is substituted for the 1/3 to get 2/6 + 1/6 = 3/6 or 1/2. Other examples such as 1/4 + 1/6 ; 1/4 + 1/3 ; and 1/5 + 1/3 may be shown by using the same procedure just described.

A third way of showing how to find a common denominator of unlike fractions is to use picture diagrams. In Figure 31, 1/2 is shown to be equal to 2/4. So 1/2 + 1/4 = 2/4 + 1/4 = 3/4. The example worked in Figure 31 is merely representative of many other similar exercises that can be solved with the help of pictures.

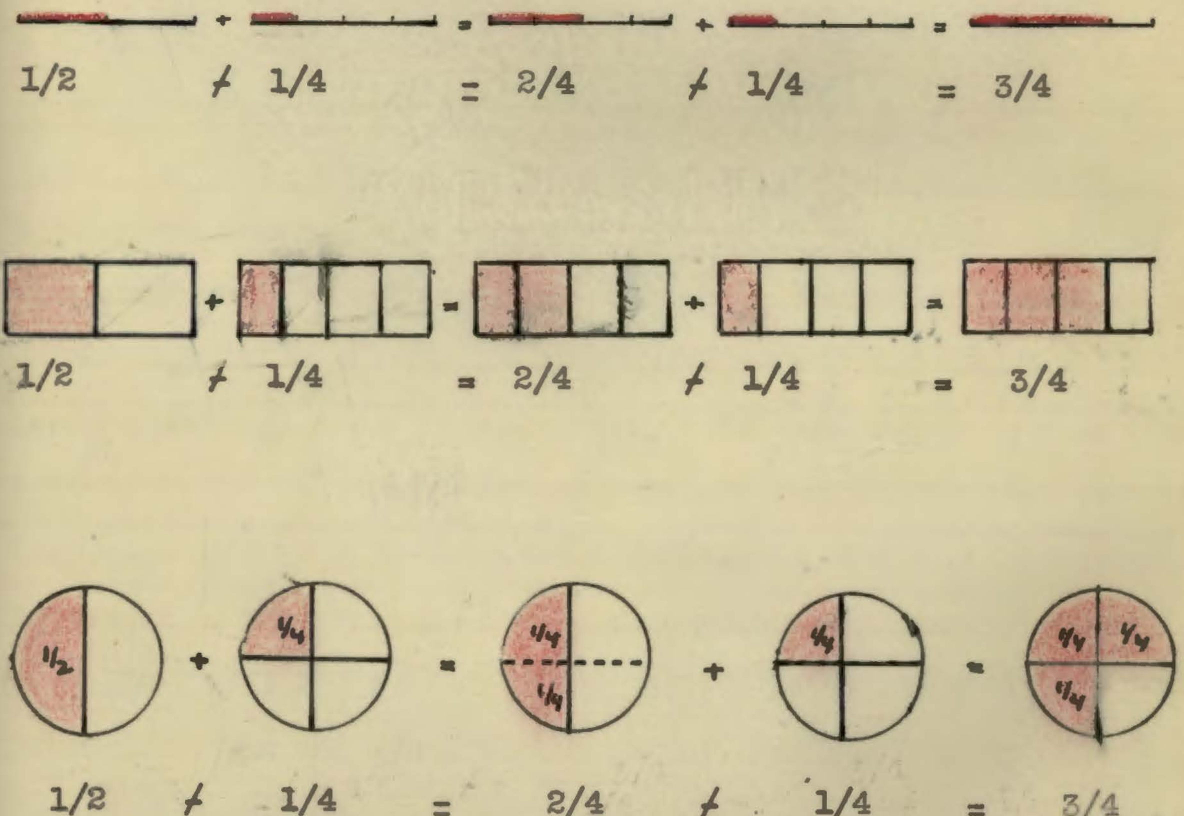



Figure 31

The use of measuring cups with colored water, and of rulers marked into fractional parts may also be used to help the students to understand how to find common denominators of unlike fractions. Two measuring cups can be used. One is filled 1/2 full of water while the other is filled 1/4 full. From the markings on the cup it can be seen that 1/2 = 2/4. Thus 1/2 + 1/4 = 2/4 + 1/4 = 3/4. By pouring the water from the two cups into one cup, the students can see that there is 3/4 of a cup of water. The students can use a ruler to find that 1/2 inch is equal to 2/4 inch. Thus 1/2 inch + 1/4 inch = 2/4 inch + 1/4 inch = 3/4 inch.

Through participation in many activities similar to those just described, the students should be able to formulate three generalizations that will help them find a common denominator for unlike fractions. These generalizations are to be formulated by the pupils, not given by the teacher as meaningless rules to be memorized. The generalizations are as follows: (1) A common denominator of unlike fractions may be the largest of the numbers that are denominators of the unlike fractions. In 1/2 + 1/4, 4 is the larger of the two numbers that are denominators and is the common denominator. (2) The product of all the denominators of the fractions is always a common denominator. It may not be the smallest common denominator, however. In 1/4 + 1/3, 12 is a common denominator and is the product of 3 x 4. (3) A multiple of a number in the denominators of the unlike fractions may be a common denominator. Twelve (6 x 2 or 4 x 3) is a common denominator for $1/4 + 1/6$.

The lowest common denominator is the smallest common denominator into which unlike fractions can be changed. In calling the student's attention to the smallest common denominator, the teacher need not demand its use. The reason many teachers emphasize the use of a lowest common denominator is that smaller figures are involved. For instance:

$$3/4 + 5/8 = 6/8 + 5/8 = 11/8 = 1\,\frac{3}{8}$$

$$3/4 + 5/8 = 24/32 + 20/32 = 44/32 = 1\,\frac{12}{32} = 1\,\frac{3}{8}$$

Both examples are correct. However 8 is smaller than 32 and thus probably is the easier of the two figures to use. Unless the teacher is stressing the concept of reducing an answer to lowest terms, the answer $1\,\frac{12}{32}$ could be counted as correct even though it is not in most simple form.

## II. UNDERSTANDING THE SUBTRACTION OF FRACTIONS

Many of the concepts learned in the addition of fractions are applicable in understanding the subtraction of fractions. For example, when like fractions are added, the numerators are added but the denominators are not. See pages 25-28 for an explanation of this concept. Part of this concept can be applied in subtracting like fractions. The difference being that we subtract instead of add the numerators of like fractions. Just as unlike fractions cannot be added until they are changed to like fractions, so unlike fractions cannot be subtracted until they are changed to like fractions. For an explanation of how unlike fractions are changed to like fractions, see pages 28-32. Thus, if the process of adding fractions is understood, the subtraction process is simple because nearly all of the concepts involved in the addition process are basic also to the subtraction process.

However, one concept used in subtracting fractions is not used in adding fractions. This concept is necessary in order to understand how to solve a subtraction problem involving borrowing. An example of such a problem is $12\frac{1}{4} - 3\frac{3}{4}$. Three-fourths cannot be taken from 1/4. There are two methods of solving this type of problem. One method is the improper fraction method and the other is the borrowing method.

<u>Improper fraction method.</u> When $1\frac{3}{4}$ is subtracted from $3\frac{1}{4}$, the 3/4 cannot be taken from the 1/4 as it stands.

$$\begin{array}{rcl} 3\frac{1}{4} & = & \frac{13}{4} \\ 1\frac{3}{4} & = & \frac{7}{4} \\ \hline & & \frac{6}{4} = 1\frac{2}{4} \text{ or } 1\frac{1}{2} \end{array}$$

Figure 32

This problem, however, can be solved by changing the mixed numbers to improper fractions. See Figure 32, page 43. The improper fraction 7/4 can be taken from 13/4 leaving 6/4. The answer, 6/4 can be changed to its simpler form of $1\frac{2}{4}$ or $1\frac{1}{2}$.

The mathematical process just described in subtracting $1\frac{3}{4}$ from $3\frac{1}{4}$ can be shown concretely with cardboard and rubber bands. As illustrated in Figure 33, four boards can be used to show $3\frac{1}{4}$ and two boards can be used to show $1\frac{3}{4}$. Each

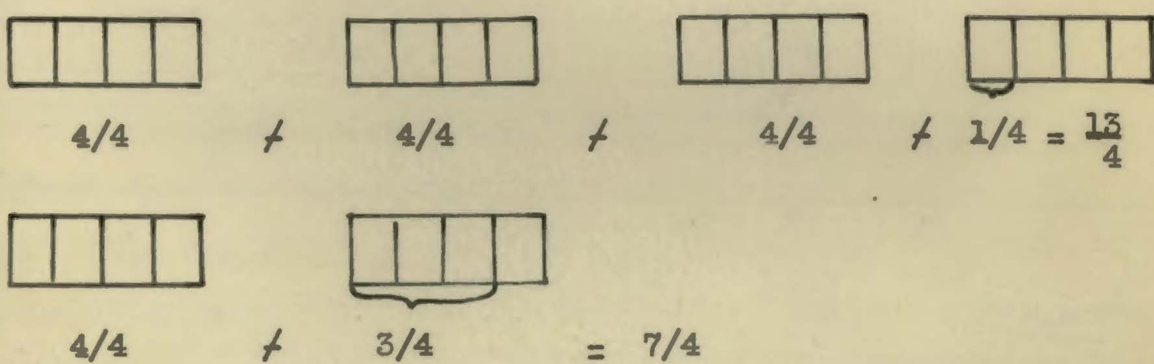



Figure 33

board is divided into fourths. The pupils can see by counting that $3\frac{1}{4}$ boards is $\frac{13}{4}$ boards and that $1\frac{3}{4}$ boards is 7/4 boards. Thus $3\frac{1}{4} = \frac{13}{4}$ and $1\frac{3}{4} = \frac{7}{4}$. Also from the boards showing 13/4, the pupils can see that only 6/4 are left when 7/4 is taken from 13/4. The red fourths in Figure 34 on page 45 are the 7/4 which are taken from the 13/4. The 6/4 left is equal to one whole board plus 2/4 or 1/2 of another board. Thus the

6/4, which is the answer to the problem shown in Figure 32, on page 43, is the same as $1\frac{1}{2}$.

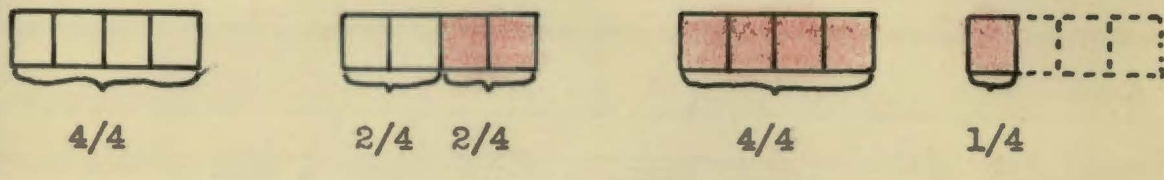



Figure 34

For a more complete discussion of how mixed numbers are changed to improper fractions, and how improper fractions are changed to mixed numbers, see pages 17-23.

<u>The borrowing method.</u> Perhaps the word "borrow" should not be used in the subtraction process because borrow implies that whatever is borrowed will be returned. In subtraction, nothing is returned. But because of the simplicity of wording and because of the general use, the term "borrow" will be used in this paper.

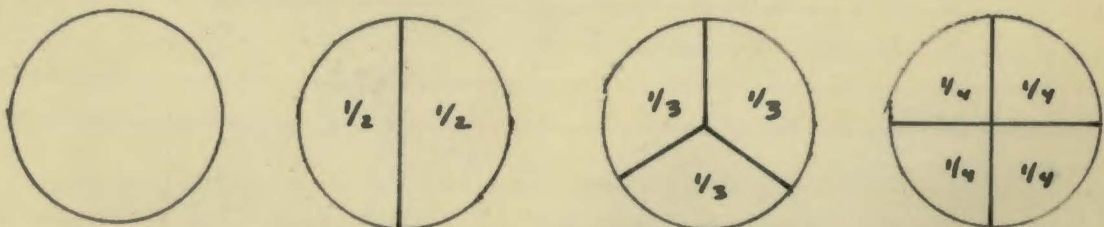
A review of the borrowing concept in whole numbers may help the student understand the same concept in fractions. Take the example of 432 - 143 shown in Figure 35. Because 3 cannot be subtracted from 2, 1 ten is changed to 10 ones and added to the 2 ones already in the one's column to get 12 ones. Three is then subtracted from 12 ones.

$$\begin{array}{r} 432 \\ -\ 143 \\ \hline 289 \end{array}$$

Figure 35

Since 1 of the 3 tens has already been used, there are just 2 tens left. But because 4 tens cannot be subtracted from 2 tens, 1 of the 4 hundreds is changed to tens. Four tens are now subtracted from 12 tens to get 8 tens. Only 3 hundreds are left after 1 hundred was changed to tens, so 1 hundred is subtracted from 3 hundred to get 2 hundred.

The concept of borrowing in subtraction of whole numbers may be applied to fractions. The teacher can first show that one whole is equal to 2/2, 3/3, and so on. The concept of changing one whole into an improper fraction with equal numerators and denominators may be made meaningful by considering the meaning of the denominator and the numerator. If a whole is divided into 2 equal parts, 2 parts are all that can be used. (See Figure 36) Any whole is equal to an improper fraction with numerator and denominator equal.




1 = 2/2 = 3/3 = 4/4

Figure 36

Consideration can now be given to examples involving the subtraction of a proper fraction from 1, such as 1 - 2/3.

An illustration of this problem is given in Figure 37. Two-thirds cannot be subtracted from 0/3, so the 1 is changed to 3/3. Two-thirds from 3/3 = 1/3. The whole number should be changed to an improper fraction with the same sized denominator as the fraction being subtracted.

$$\begin{array}{rrcr} & 1 & = & 3/3 \\ - & 2/3 & = & 2/3 \\ \hline & & & 1/3 \end{array}$$

Figure 37

If 1/4 is subtracted from 1, the 1 is changed to 4/4, but if 5/6 is subtracted, the 1 is changed to 6/6.

When the student understands that a whole number can be changed to an improper fraction, examples that have a fraction subtracted from any whole number may be used. Figure 38 shows the problem, 5 - 2/3. Two-thirds cannot be subtracted from 0/3, so 1 of the 5 ones is changed to 3/3. That leaves 4 ones and 3/3, or $4\frac{3}{3}$. Two-thirds from $4\frac{3}{3} = 4\frac{1}{3}$. A similar principal is involved when the whole numbers, 34 - 16, are subtracted as shown in Figure 39. Six ones cannot be taken from 4 ones, so 1 of the 3 tens is changed to 10 ones, leaving 2 tens. The 10 ones are added to the 4 ones already in the ones' column to get 14 ones. Now 14 ones - 6 ones = 8 ones. One ten from 2 tens leaves 1 ten. Thus 34 - 16 = 18.

$$\begin{array}{rrcr} & 5 & = & 4\frac{3}{3} \\ - & 2/3 & = & \frac{2}{3} \\ \hline & & & 4\frac{1}{3} \end{array}$$

Figure 38

$$\begin{array}{r} 34 \\ -\ 16 \\ \hline 18 \end{array}$$

Figure 39

The final and most difficult step in the subtraction of fractions involves examples in which fractions are subtracted from mixed numbers and borrowing is necessary. In Figure 40, $6\frac{1}{4} - 3/4$ is shown.

$$\begin{array}{rcl} 6\frac{1}{4} & = & 5\frac{5}{4} \\ -\quad \frac{3}{4} & = & \frac{3}{4} \\ \hline & & 5\frac{2}{4} = 5\frac{1}{2} \end{array}$$

Figure 40

Borrowing is necessary because 3/4 cannot be subtracted from 1/4. One of the 6 ones is changed to 4/4. That leaves 5 ones, while the 4/4 is added to the 1/4 to get 5/4. Thus $6\frac{1}{4} = 5\frac{5}{4}$. Now 3/4 is taken from $5\frac{5}{4}$ to leave $5\frac{2}{4}$ or $5\frac{1}{2}$.

Both the improper fraction method and the borrowing method in subtraction of fractions have their advantages and disadvantages. The borrowing method usually involves smaller figures than does the improper fraction method and thus is perhaps simpler to use. In the example $12\frac{1}{4} - 3\frac{3}{4}$, smaller figures are involved when the problem is solved by the borrowing method. See Figure 41.

$$\begin{array}{rcl} 12\frac{1}{4} & = & 11\frac{5}{4} \\ -\ 3\frac{3}{4} & = & 3\frac{3}{4} \\ \hline & & 8\frac{2}{4} = 8\frac{1}{2} \end{array}$$

Borrowing Method

$$\begin{array}{rcl} 12\frac{1}{4} & = & \frac{49}{4} \\ -\ 3\frac{3}{4} & = & \frac{15}{4} \\ \hline & & \frac{34}{4} = 8\frac{2}{4} = 8\frac{1}{2} \end{array}$$

Improper Fraction Method

Figure 41

In Chapter II, some of the basic concepts in the addition and subtraction of fractions have been considered. These concepts are to be made meaningful to the student through actual participation in concrete, semi-concrete and abstract activities.

# CHAPTER III

## UNDERSTANDING THE MULTIPLICATION OF FRACTIONS

Chapter III deals with understanding the fundamental process of multiplication of fractions. The mechanics of the multiplication of fractions are relatively simple but many students do not understand what actually happens when these simple mechanical operations are used. The purpose of this chapter is to develop meaningfully some basic concepts necessary to understand the multiplication of fractions.

### I. MULTIPLICATION WHEN ONE FACTOR IS A WHOLE NUMBER

<u>The multiplication of a fraction by a whole number.</u> The solution to the multiplication of a fraction by a whole number is based upon the concept of multiplication of whole numbers. Four x 15 means that four 15's are to be added. (15 + 15 + 15 + 15 = 60.) A review of the meaning of multiplication of whole numbers may be beneficial. As shown in

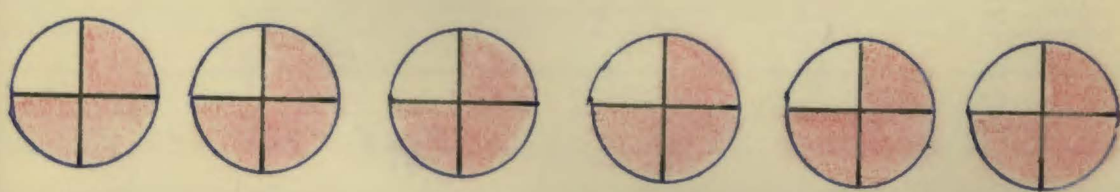
| | | |
|---|---|---|
| 5 x 4 = 20 | or | 4 + 4 + 4 + 4 + 4 = 20 |
| 2 x 14 = 28 | or | 14 + 14 = 28 |
| 10 x 12 = 120 | or | 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 = 120 |

Figure 42

Figure 42, problems such as 5 x 4, 2 x 14, 10 x 12, and so on may be solved by both addition and multiplication. Five x 4 means that 4 is added 5 times. Two x 14 means that 14 is added two times, and 10 x 12 means that ten 12's are to be added. From this activity the students should see that a multiplication problem may be solved by addition. Thus, 6 x 3/4 means that 3/4 is added 6 times to get 18/4. (3/4 + 3/4 + 3/4 + 3/4 + 3/4 + 3/4 = 18/4.) The answer 18/4 may be changed to the mixed number of $4\frac{1}{2}$. Several problems involving the multiplication of a fraction by a whole number should be solved by the addition method. Now the teacher can have the pupils look at these problems which they have solved by the addition method to find a rule which they could use in solving the problem by multiplication. When 6 x 3/4 is solved by the addition method, the answer is found to be 18/4. Therefore, 6 x 3/4 = 18/4. The students should be able to discover that 6 x 3 = 18 and the denominator is not changed. However, the child should not be rushed. If he does not discover the rule, more of the same work may help or another method of presentation may be used. But the teacher should avoid giving the student the rule to be memorized until he understands the concept.

Another method of presentation might be to change the denominator to a word. The fraction 3/4 may be written as 3 fourths. Now 6 x 3/4 = 6 x 3 fourths. Fourths may be considered as a unit of measure just as a foot, a yard, or an hour. For a detailed discussion of the concept of the denominator as a unit of measure, the reader is referred to pages 13-16 and 25-26. Just as 6 x 3 feet = 18 feet, so 6 x 3 fourths = 18 fourths. Eighteen-fourths can be written as 18/4. Thus 6 x 3 fourths = 18 fourths which can be written 6 x 3/4 = 18/4. The student should be allowed to solve several such examples in order to discover that the numerator of a fraction is multiplied by the whole number to get the numerator of the answer and that the denominator is not changed. In the problem 6 x 3/4 = 18/4, the 18 is arrived at by multiplying 6 x 3 and the denominator is not changed.

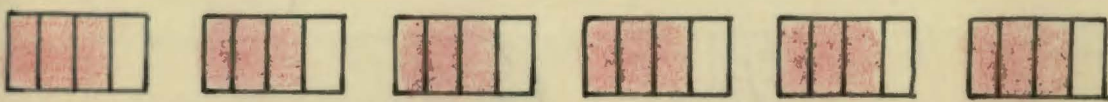
The use of diagrams is still another way of presenting the process of the multiplication of a fraction by a whole number. In Figure 43 the number of colored fourths may be counted to find 18/4. Cardboards and rubber bands



6 x 3/4 = 3/4 + 3/4 + 3/4 + 3/4 + 3/4 + 3/4 = 18/4

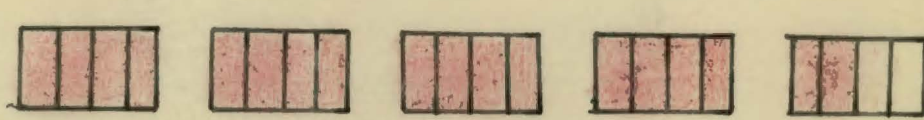
Figure 43

may also be used to show the student what happens when 3/4 is multiplied by 6. Six cardboards may be divided into fourths. Three-fourths of each board is used. The pupils may count how many fourths are used on all six boards. They of course will find that 18/4 are used altogether. Next the pupils can find how many whole boards and what fractional part of a board is used in 18/4. Figure 44 shows that 6 x 3/4 = 18/4 = 4 $\frac{2}{4}$ = 4 $\frac{1}{2}$ .



6 x 3/4 = 3/4 + 3/4 + 3/4 + 3/4 + 3/4 + 3/4 =

18/4



=

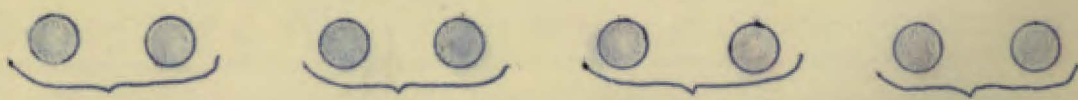
4 $\frac{2}{4}$ or 4 $\frac{1}{2}$

Figure 44

By participating in addition activities to solve multiplication problems, by using a cardboard and rubber bands to show what happens in multiplication of fractions, by using diagrams to picture the multiplication process, and by changing the denominator of the fraction to a written word, the student should be able to discover that the numerator of the fraction is multiplied by the whole number and the denominator is not changed.

Multiplying a whole number by a fraction. The mechanics of multiplying a whole number by a fraction is relatively simple to understand. The answer to 3/4 x 8 is the same as 8 x 3/4. Most students know that 5 x 4 = 4 x 5 and will be able to see that 3/4 x 8 = 8 x 3/4. The numerator of the fraction times the whole number equals the numerator of the answer and the denominator is not changed. (3/4 x 8 = 24/4 = 6.)

However, the meaning of 3/4 x 8 is not the same as 8 x 3/4. Many students may be able to find the answer to 3/4 x 8 without understanding what the problem means. The problem does not mean that 8 is added 3/4 of a time. To solve this by addition is impossible because 8 cannot be added 3/4 of a time. Three-fourths of 8, means that 8 is divided into 4 equal parts and 3 of those 4 equal parts are being used. In solving the problem, the students should first consider what 1/4 of 8 equals. As shown in Figure 45, 1/4 of 8 = 2.



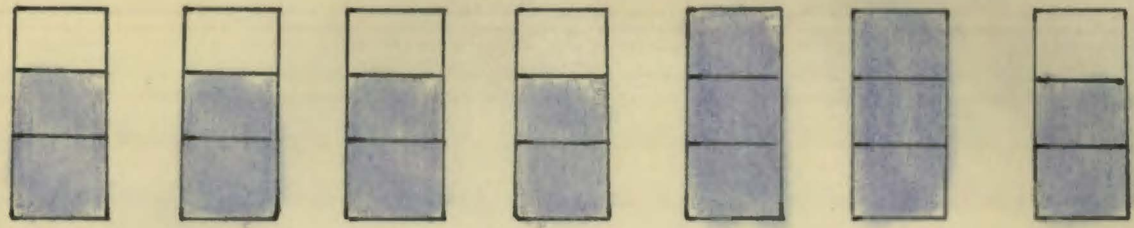
1/4 of 8 = 2 1/4 of 8 = 2 1/4 of 8 = 2 1/4 of 8 = 2

Figure 45

Three-fourths means that 3 of the 4 equal parts are being used, so 3/4 means 3 x 1/4. Since 1/4 of 8 = 2, 3/4 of 8 = 3 x 2 = 6.

The concept of finding a fractional part of a whole number should not be given to the students without activities which help to increase the meaning. Eight marbles may be divided into 4 equal groups. Each group will have 2 marbles in it. Three of the groups will be equal to 3 x 2 or 6. Pictures such as shown in Figure 45, page 54, help the student understand that 3/4 of 8 = 6. Many examples need to be explained by using concrete objects or pictures to show the meaning of a fraction of a whole number. The denominator of the fraction tells us the number of equal parts into which the whole number is divided. The size of each equal group is multiplied by the numerator of the fraction. Thus 5/6 of 12 means that 12 is divided into 6 equal groups. Each equal group is equal to 2. Five x 2 = 10.

Examples such as 2/3 of 4 are difficult to explain by the use of concrete objects or by picture diagrams because each equal group is not equal to a whole number as in 3/4 of 8. Cardboards with rubber bands may be used to illustrate 2/3 of 4. In Figure 46, page 56, each of 4 boards is divided into 3 equal parts, and 2 of the 3 equal parts of each board are colored. The students can then count the thirds on their boards to determine the answer. One board illustrates 2 of the 3 equal parts, and all four boards illustrate 8 thirds. From the boards the student can also discover $8/3 = 2\frac{2}{3}$.



$\frac{2}{3}$ of 1 $\frac{2}{3}$ of 1 $\frac{2}{3}$ of 1 $\frac{2}{3}$ of 1 = 8/3 = $2\frac{2}{3}$

2/3 of 4 = 8/3 = $2\frac{2}{3}$

Figure 46

After solving several similar exercises by using the cardboards, the teacher can have the students see if they can discover a mathematical rule for finding 2/3 of a whole number. The students should be able to discover that the numerator of the fraction is multiplied by the whole number and the denominator is not changed. That is, 2/3 of 4 = 8/3. By participating in meaningful activities the student will be able to understand the meaning back of the mechanical process of multiplying a whole number by a fraction. Rules discovered by students after participating in activities are meaningful, while those given by the teacher to be memorized are frequently completely devoid of meaning.

II. MULTIPLICATION WHEN BOTH FACTORS ARE FRACTIONS

The meaning of multiplying a fraction by a fraction is not understood by many students because they have not participated in activities which will help them develop meaning. Many teachers merely tell the student to multiply the numerators together to get a new numerator and to multiply the denominators together to get a new denominator. Frequently, no attempt is made to explain the meaning back of the mechanical process. The student is asked to memorize a meaningless rule.

Activities that help the student understand multiplication of a fraction times a fraction. Cardboards and rubber bands can be used to help students understand the meaning of multiplying a fraction by a fraction. Take 1/2 x 1/2, for instance. From activities and experiences in the multiplication of a whole number by a fraction, the student has learned that the word "of" means to multiply. Therefore 1/2 x 1/2 means 1/2 of 1/2. A cardboard can be divided into 2/2 with a red rubber band as in Figure 47.

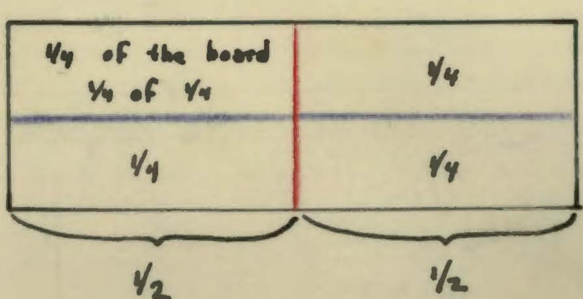



Figure 47

Then 1/2 of 1/2 can be shown by placing a blue rubber band the other direction on the board. The fact that 1/2 of 1/2 is equal to 1/4 of the whole board can then be seen clearly. Figure 48 shows that 1/2 of 3/4 = 3/8. The board is divided into fourths with red bands, each fourth is then divided in half by placing a blue band the other direction on the board. Many other examples similar to those shown in Figures 47 on page 57 and Figure 48 given above, should be worked by the student. After the students solve these exercises on their boards, the teacher can have them write down all the exercises and answers. For example, 1/2 of 3/4 = 1/2 x 3/4 = 3/8. By inspection, the rule for multiplying a fraction by a fraction can be discovered. The numerators are multiplied together to get a new numerator and the denominators are multiplied together to get a new denominator.

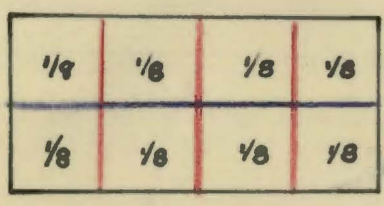



Figure 48

Figure 49 illustrates how students can use their boards to verify the rule they discovered. For example, to determine that

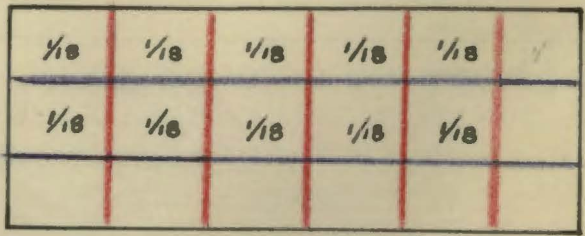



Figure 49

$\frac{2}{3} \times \frac{5}{6} = \frac{10}{18} - \frac{5}{9}$ , the students can use red rubber bands to divide the board into sixths. Each sixth is then divided into 3 equal parts by 2 blue bands placed horizontally. The student can now count to discover that $\frac{2}{3}$ of $\frac{5}{6} = \frac{10}{18}$ . Ten-eighteenths can be reduced to 5/9. The verification of the rule for multiplication of a fraction by a fraction by using the cardboards and rubber bands is a very good review technique.

## III. MULTIPLICATION OF MIXED NUMBERS

Multiplication by changing a mixed number to an improper fraction. As explained on pages 20-22, and on pages 44-45, a mixed number such as $2\frac{2}{3}$ may be changed to 8/3, an improper fraction. Two is equal to 6/3. Six-thirds ≠ 2/3 = 8/3. Any multiplication problem involving a mixed number may be solved by changing the mixed number to an improper fraction and then multiplying the improper fraction like a proper fraction, as was explained on pages 56-59. Figure 50 shows three types of problems involving mixed numbers--a whole number times a mixed number, a mixed number times a whole number, and a mixed number times a mixed number.

(1) $5 \times 2\frac{2}{3} = 5 \times 8/3 = 40/3 = 13\frac{1}{3}$

(2) $4\frac{1}{5} \times 6 = 21/5 \times 6 = 126/5 = 25\frac{1}{5}$

(3) $1\frac{3}{4} \times 3\frac{5}{6} = 7/4 \times 23/6 = 161/24 = 6\frac{17}{24}$

Figure 50

<u>Multiplication by using the whole number.</u> In some exercises involving a whole number times a mixed number or a mixed number times a whole number, the whole number form of multiplication may be simpler to use than it would be to change the mixed number to an improper fraction. As is illustrated in Figure 51, 15 x 42 is worked by first multiplying 5 x 42 to get 210, and then multiplying 10 x 42 to get 420. These two partial products, 210 and 420, are added to get 630. Fifteen is equal to 10 + 5. Therefore 15 x 42 = 5 x 42 + 10 x 42.

$$\begin{array}{r} 42 \\ \underline{\times\ 15} \\ 210 \\ \underline{420} \\ 630 \end{array}$$

Figure 51

The same concept involved in the multiplication of whole numbers is applicable to the multiplication of a mixed number by a whole number.

In the exercise $8 \times 4\frac{3}{4}$, see Figure 52, 8 x 3/4 = 6, and 8 x 4 = 32. Four and three-fourths = 4 + 3/4. Eight x $4\frac{3}{4}$ = 8 x 3/4 + 8 x 4. Thus $8 \times 4\frac{3}{4} = 6 +$ 32 = 38. The multiplication of a whole number by a mixed number can also be worked using the same form. For example, see Figure 53, page 61, which shows $7\frac{1}{2} \times 14$. Seven and one-

$$\begin{array}{r} 4\frac{3}{4} \\ \underline{\times\ 8} \\ 6 \\ \underline{32} \\ 38 \end{array} \qquad \begin{array}{l} 8 \times 3/4 = \\ 24/4 = 6 \end{array}$$

Figure 52

half x 14 means 1/2 x 14 plus 7 x 14 = 7 + 98 = 105.

Not all problems involving mixed numbers need to be solved by the same mechanical process. In some it is easier to change the mixed number to an improper fraction and then multiply like any fraction problem. In other examples, it is easier to multiply using the form used in multiplication of whole numbers. Whichever process is the simpler and easier is the best process to use.

$$\begin{array}{r} 14 \\ \times \quad 7\frac{1}{2} \\ \hline 7 \\ 98 \\ \hline 105 \end{array} \qquad 1/2 \times 14 = 7$$

Figure 53

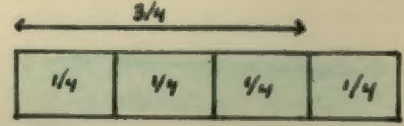
Chapter III, now being concluded, has presented activities which the teacher can employ to help students develop the meanings that are necessary to understand the multiplication of fractions. The meanings back of the relatively simple mechanical procedures must be developed through pupil participation in meaningful activities. From these activities, the student should be able to discover the rules for multiplication of fractions.

## CHAPTER IV

## UNDERSTANDING THE DIVISION OF FRACTIONS

Chapter IV explains how the division of fractions can be made meaningful to students in the sixth grade. Many students fail to understand the division of fractions because their teachers have failed to present the division of fractions meaningfully. The rule "invert the divisor and multiply" is mechanical in nature and does not give the students any understanding of the process of division. The meaning of the division of fractions can be developed by using diagrams, cardboards with rubber bands, and other concrete materials. Then, after the student understands the division of fractions, the teacher may present the "inversion method" as a short cut to get the answer.

In solving the problem $3/4 \div 1/4$, a diagram similar to Figure 54 may be used. The problem means how many 1/4 size pieces are found in 3/4. From the diagram, the student can count three 1/4's in 3/4. Thus $3/4 \div 1/4 = 3$. The same type of diagram as shown in Figure 54 may be used in solving the following: $2/3 \div 1/3 = 2$; $5/8 \div 1/8 = 5$; $6/8 \div 3/8 = 2$; and $3/16 \div 1/16 = 3$.




$3/4 \div 1/4 = 3$

Figure 54

After solving several problems similar to 3/4 ÷ 1/4 which have a whole number for a quotient, problems having a mixed number in the quotient may be introduced. The problem, 5/8 ÷ 2/8 means how many 2/8 size pieces are found in 5/8. From the diagram in Figure 55, the student can count two

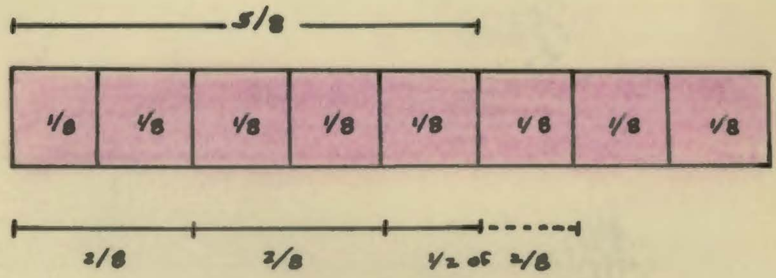



Figure 55

2/8 pieces and 1/2 of another 2/8 piece in 5/8. Therefore, 5/8 ÷ 2/8 = $2\frac{1}{2}$. Similar diagrams may be used to solve 6/8 ÷ 4/8 = $1\frac{1}{2}$; 3/4 ÷ 2/4 = $1\frac{1}{2}$; 7/10 ÷ 3/10 = $2\frac{1}{3}$; and $\frac{12}{16}$ ÷ $\frac{5}{16}$ = $2\frac{2}{5}$.

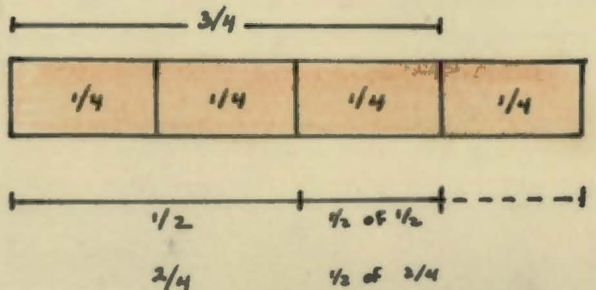
When several problems involving like fractions have been solved by using diagrams like Figure 54, page 62, and Figure 55, the problems with their answers can be written down by the pupils as follows:

| | |
|---|---|
| 3/4 ÷ 1/4 = 3 | 5/8 ÷ 2/8 = $2\frac{1}{2}$ |
| 2/3 ÷ 1/3 = 2 | 6/8 ÷ 4/8 = $1\frac{2}{4}$ or $1\frac{1}{2}$ |
| 5/8 ÷ 1/8 = 5 | 3/4 ÷ 2/4 = $1\frac{2}{4}$ or $1\frac{1}{2}$ |
| 6/8 ÷ 3/8 = 2 | 7/10 ÷ 3/10 = $2\frac{1}{3}$ |
| 2/16 ÷ 1/16 = 2 | 12/16 ÷ 5/16 = $2\frac{2}{5}$ |

The students can then study the division problems with their answers to find a mathematical solution when like fractions are divided. With the problems listed, the students should discover that the answer is found by dividing the numerator of the dividend by the numerator of the divisor as is shown in the following problems:

$$3/4 \div 1/4 = 3 \div 1 = 3$$
$$2/3 \div 1/3 = 2 \div 1 = 2$$
$$5/8 \div 1/8 = 5 \div 1 = 5$$
$$6/8 \div 3/8 = 6 \div 3 = 2$$
$$3/16 \div 1/16 = 3 \div 1 = 3$$
$$5/8 \div 2/8 = 5 \div 2 = 2\frac{1}{2}$$
$$6/8 \div 4/8 = 6 \div 4 = 1\frac{2}{4} \text{ or } 1\frac{1}{2}$$
$$3/4 \div 2/4 = 3 \div 2 = 1\frac{1}{2}$$
$$7/16 \div 3/16 = 7 \div 3 = 2\frac{1}{3}$$
$$12/16 \div 5/16 = 12 \div 5 = 2\frac{2}{5}$$

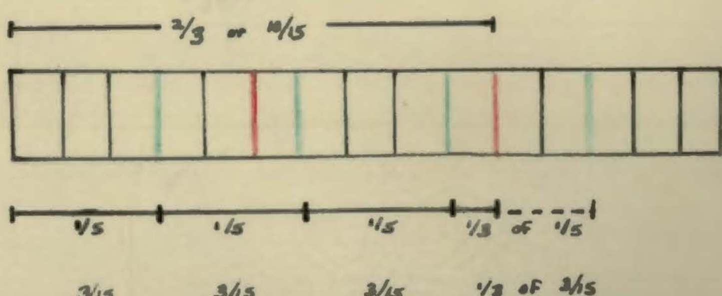
Problems involving unlike fractions when the divisor is smaller than the dividend is another type of division problem in fractions. Examples of such problems might include 3/4 ÷ 1/2; 5/8 ÷ 1/2; 6/8 ÷ 1/4; 6/8 ÷ 1/2 and 5/12 ÷ 1/6. The solution to 3/4 ÷ 1/2 is shown in Figure 58, page 65. The problem means how many 1/2 size pieces are found in 3/4. From the diagram, it can be seen that there is one 1/2 piece and 1/2 of another 1/2 piece; thus 3/4 ÷ 1/2 = $1\frac{1}{2}$ . The same diagram may be used to show that $3/4 \div 1/2 = 3/4 \div 2/4 = 1\frac{1}{2}$. Thus to solve mathematically $3/4 \div 1/2$, the fractions are changed to like fractions. Then the like fractions in $3/4 \div 2/4$ are divided by dividing the numerator of the dividend by the numerator of the divisor ($3/4 \div 1/2 = 3/4 \div 2/4 = 3 \div 2 = 1\frac{1}{2}$.)




$$3/4 \div 1/2 = 3/4 \div 2/4 = 3 \div 2 = 1\tfrac{1}{2}$$

Figure 56

Shown in Figure 57 is the diagram solution to $2/3 \div 1/5$. In order to find how many 1/5's are in 2/3, a common denominator is found. The 2/3 and the 1/5 are changed to 15ths




$$2/3 \div 1/5 = 10/15 \div 3/15 = 10 \div 3 = 3\tfrac{1}{3}$$

Figure 57

so that $2/3 \div 1/5 = 10/15 \div 3/15$. The problem now becomes how many $3/15$ are found in $10/15$. When $10/15$ is divided by $3/15$, the answer is found mathematically by dividing the numerator of the dividend by the numerator of the divisor. Thus $2/3 \div 1/5 = 10/15 \div 3/15 = 10 \div 3 = 3\frac{1}{3}$.

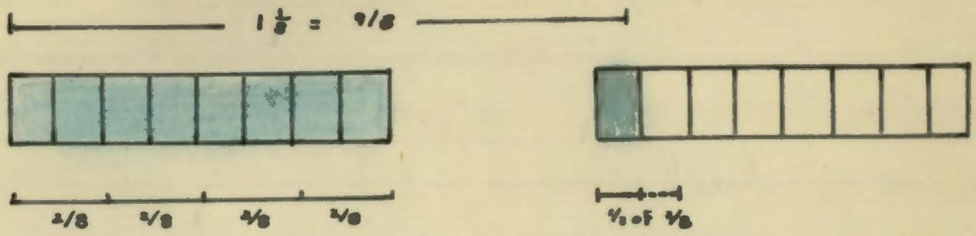
Other diagrams similar to Figures 36 and 37 on page 65 can be used to solve $6/8 \div 1/4 = 3$; $6/8 \div 1/2 = 1\frac{1}{2}$; and $5/12 \div 1/6 = 2\frac{1}{2}$. By solving these problems with diagrams just as $3/4 \div 1/2$ and $2/3 \div 1/5$ were solved, the following list can be made by the pupils:

$$3/4 \div 1/2 = 3/4 \div 2/4 = 3 \div 2 = 1\frac{1}{2}$$
$$5/8 \div 1/2 = 5/8 \div 4/8 = 5 \div 4 = 1\frac{1}{4}$$
$$6/8 \div 1/4 = 6/8 \div 2/8 = 6 \div 2 = 3$$
$$6/8 \div 1/2 = 6/8 \div 4/8 = 6 \div 4 = 1\frac{2}{4} \text{ or } 1\frac{1}{2}$$

By studying this list, the student should see that the unlike fractions can be changed to like fractions with a common denominator and then solved by dividing the numerator of the dividend by the numerator of the divisor. In the problem, $3/4 \div 1/2$, the fractions are changed to the common denominator of 4 to get $3/4 \div 1/2 = 3/4 \div 2/4$. Then 3 is divided by 2 to get $1\frac{1}{2}$.

When dividing a mixed number by a proper fraction, a diagram can be used to show the solution such as is done in

Figure 58 where $1\frac{1}{8}$ is divided by 2/8. The problem means how many 2/8 pieces are found in $1\frac{1}{8}$. Since $1\frac{1}{8} = 9/8$ as shown

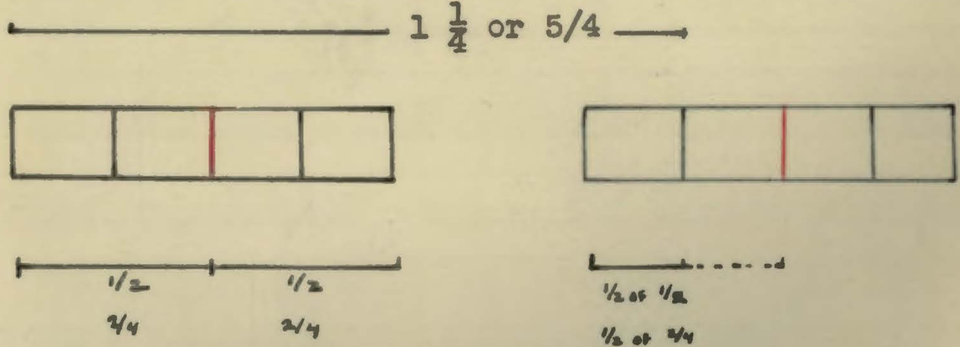



$$1\frac{1}{8} \div 2/8 = 9/8 \div 2/8 = 9 \div 2 = 2\frac{1}{2}$$

Figure 58

in the Figure, the problem becomes how many 2/8 pieces are found in 9/8. From the diagram, the pupil can find by counting that there are four 2/8 pieces and 1/2 of another 2/8 piece in 9/8; thus $1\frac{1}{8} \div 2/8 = 9/8 \div 2/8 = 9 \div 2 = 4\frac{1}{2}$.

The problem $1\frac{1}{4} \div 1/2$ is pictured in Figure 59. The $1\frac{1}{4}$ is the same as 5/4. Thus $1\frac{1}{4} \div 1/2 = 5/4 \div 1/2$. How




$$1\frac{1}{4} \div 1/2 = 5/4 \div 1/2 = 5/4 \div 2/4 = 5 \div 2 = 2\frac{1}{2}$$

Figure 59

many 1/2's are in 5/4? By counting, the pupil can find that there are two 1/2 pieces and 1/2 of another 1/2 piece in 5/4. The same figure shows that $1/2 = 2/4$, and that $1\frac{1}{4} \div 1/2 = 5/4 \div 1/2 = 5/4 \div 2/4 = 5 \div 2 = 2\frac{1}{2}$.

Other problems such as $1\frac{1}{6} \div 2/3$ and $4\frac{1}{2} \div 3/4$, can be solved by diagrams similar to those in Figures 58 and 59, page 67. By studying the solutions from the diagrams, the following mathematical solutions can be discovered:

$$1\frac{1}{8} \div 2/8 = 9/8 \div 2/8 = 9 \div 2 = 4\frac{1}{2}$$

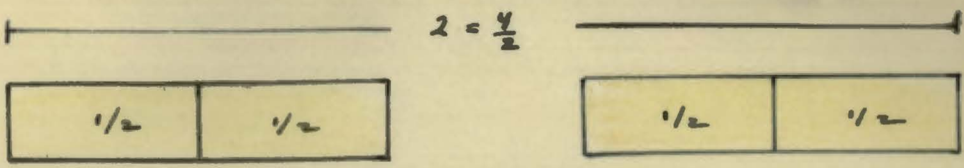
$$1\frac{1}{4} \div 1/2 = 5/4 \div 1/2 = 5/4 \div 2/4 = 5 \div 2 = 2\frac{1}{2}$$

$$1\frac{1}{6} \div 2/3 = 7/6 \div 2/3 = 7/6 \div 4/6 = 7 \div 4 = 1\frac{3}{4}$$

$$4\frac{1}{2} \div 3/4 = 9/2 \div 3/4 = \frac{18}{4} \div 3/4 = 18 \div 3 = 6$$

The mixed number is changed to an improper fraction and then a common denominator is found. The like fractions are then divided by dividing the numerator of the dividend by the numerator of the divisor.

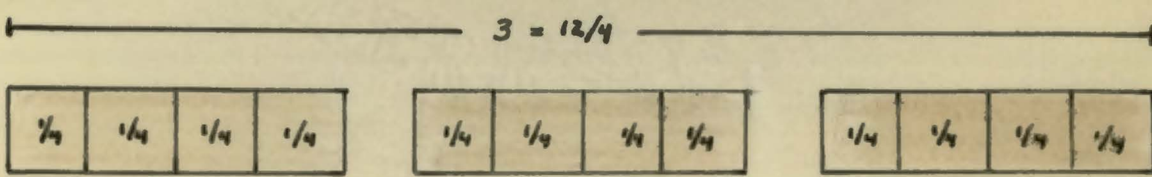
When a whole number is divided by a proper fraction, the whole number can be changed to an improper fraction with the same denominator as the divisor. For instance, 2 is equal to 4/2 as shown in Figure 60, page 69. Then $2 \div 1/2 = 4/2 \div 1/2 = 4 \div 1 = 4$.




$2 \div 1/2 = 4/2 \div 1/2 = 4 \div 1 = 4$

Figure 60

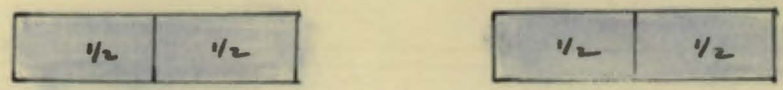
In Figure 61, 3 is divided by 1/4. The 3 wholes = $\frac{12}{4}$ . Then, $3 \div 1/4 = \frac{12}{4} \div 1/4 = 12 \div 1 = 12$.




$3 \div 1/4 = 12/4 \div 1/4 = 12 \div 1 = 12$

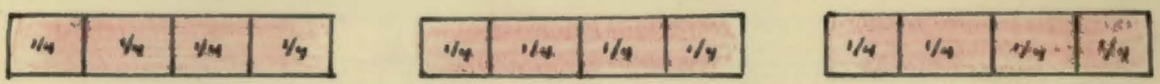
Figure 61

Another way of showing that $2 \div 1/2 = 4$ and $3 \div 1/4 = 12$ is to first consider $1 \div 1/2$ and $1 \div 1/4$. As seen in Figure 62, page 70, $1 \div 1/2 = 2$ and $1 \div 1/4 = 4$.




$1 \div 1/2 = 2$ $\qquad$ $1 \div 1/2 = 2$

$$2 \div 1/2 = 2 \times (1 \div 1/2) = 2 \times 2 = 4$$



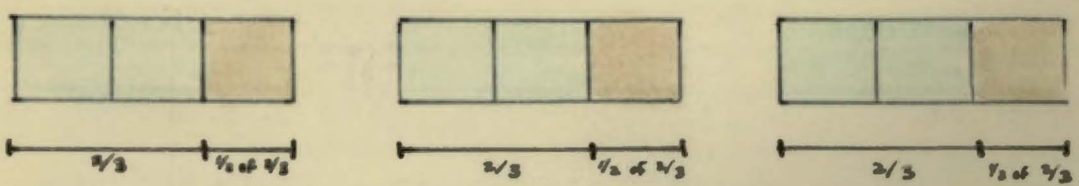

$1 \div 1/4 = 4$ $\qquad$ $1 \div 1/4 = 4$ $\qquad$ $1 \div 1/4 = 4$

$$3 \div 1/4 = 3 \times (1 \div 1/4) = 3 \times 4 = 12$$

Figure 62

Since 2 is $2 \times 1$, then $2 \div 1/2$ is $2 \times (1 \div 1/2)$ or $2 \times 2 = 4$. Similarly, 3 is $3 \times 1$, so $3 \div 1/4 = 3 \times (1 \div 1/4)$ or $3 \times 4 = 12$.

Figure 63 shows $3 \div 2/3$. First consider $1 \div 2/3$. There are, as seen in the Figure, $1\frac{1}{2}$ two-thirds in 1 whole. Since $3 = 3 \times 1$, $3 \div 2/3 = 3 \times (1 \div 2/3)$ or $3 \times 1\frac{1}{2} = 4\frac{1}{2}$.




$1 \div 2/3 = 1\frac{1}{2}$ $\qquad$ $1 \div 2/3 = 1\frac{1}{2}$ $\qquad$ $1 \div 2/3 = 1\frac{1}{2}$

$$3 \div 2/3 = 3 \times (1 \div 2/3) = 3 \times 1\frac{1}{2} = 4\frac{1}{2}$$

Figure 63

When the "inversion method" of dividing fractions is introduced, exercises involving a whole number divided by a fraction may be used to help the student see that a division problem may be solved by multiplying the dividend times the divisor inverted. For instance as can be seen in Figure 62, page 70, $2 \div 1/2 = 2 \times 2 = 4$ and $3 \div 1/4 = 3 \times 4 = 12$. Also Figure 63, page 70, shows that $3 \div 2/3 = 3 \times 1\frac{1}{2} = 3 \times 3/2 = 4\frac{1}{2}$. By studying diagrams similar to these two figures, the student should be able to understand that it is possible to solve a division problem in fractions by multiplying the dividend by the inverted divisor.

Up to this point, in Chapter IV, only problems involving the measurement concept of division have been considered. The problem $3/4 \div 1/4$ means how many 1/4's are contained in 3/4. This is a measurement concept in division. Other problems involving the same measurement concept are:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5/8 \div 2/8$ | means | how | many | 2/8's | are | found | in | 5/8 |
| $3/16 \div 1/16$ | " | " | " | 1/16's | " | " | " | 3/16 |
| $3/4 \div 1/2$ | " | " | " | 1/2's | " | " | " | 3/4 |
| $1\frac{1}{8} \div 2/8$ | " | " | " | 2/8's | " | " | " | $1\frac{1}{8}$ |
| $3 \div 1/4$ | " | " | " | 1/4's | " | " | " | 3 |

However, the measurement concept in division does not apply to problems such as $1/2 \div 2$; $2/3 \div 3$; $1/3 \div 4$; etc. To explain the meaning of these problems, the partition concept of division must be used. The partition concept applied in these problems is as follows:

> 1/2 ÷ 2 means that 1/2 is divided into 2 equal parts. What is the size of each part?
>
> 2/3 ÷ 3 means that 2/3 is divided into 3 equal parts. What is the size of each part?
>
> 1/3 ÷ 4 means that 1/3 is divided into 4 equal parts. What is the size of each part?

When solving problems involving the partition concept, diagrams, cardboards with rubber bands, and other concrete and semi-concrete material can be used to illustrate the meaning involved in the problems. Figure 64 illustrates 1/2 ÷ 2. The whole rectangle is divided into halves by the blue vertical line. Now each of these halves is divided into 2 equal parts as shown by the red horizontal line. Thus by counting, the student can find that 2/2 ÷ 2 = 2/4 or 1/2. But when 1/2 is divided into 2 equal pieces, each piece is 1/4 of the whole rectangle as shown by the blue diagonal lines. Therefore, 1/2 ÷ 2 = 1/4.

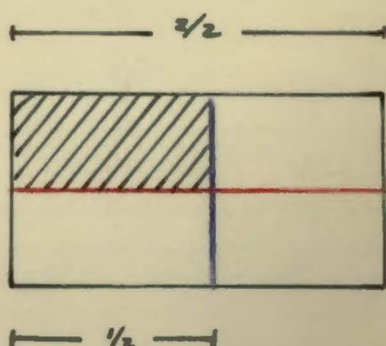



Figure 64

Diagrams similar to Figure 64 can be used to solve 2/3 ÷ 3; 1/4 ÷ 3; and any other exercise involving a proper fraction divided by a whole number.

The problem $1/3 \div 2/3$ has a divisor larger than the dividend. To help the students understand the meaning of this kind of exercise, the following division exercises involving whole numbers should be solved first:

$16 \div 2 = 8$. How many 2's are there in 16?

$16 \div 4 = 4$. How many 4's are there in 16?

$16 \div 8 = 2$. How many 8's are there in 16?

$16 \div 16 = 1$. How many 16's are there in 16?

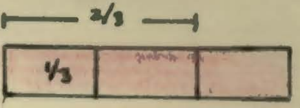
$16 \div 32 = \frac{16}{32} = 1/2$. Sixteen is what part of 32?

$16 \div 64 = \frac{16}{64} = 1/4$. Sixteen is what part of 64?

Until the divisor becomes larger than the dividend, each example asks, "how many of a certain quantity are in 16?" But when 16 is divided by 32, the meaning changes from how many 32's are in 16 to, 16 is what part of 32? Since 32 is larger than 16, there cannot be 1 whole 32 in 16. Therefore, the question becomes, 16 is what fractional part of 32. The problem, $16 \div 32$ is then solved by writing it as a fraction, $\frac{16}{32}$ which can be reduced to 1/2. Thus, 16 is 1/2 of 32.

The exercies $1/3 \div 2/3$ is similar in meaning to $16 \div 32$. Each exercise has a divisor larger than the dividend. Just as $16 \div 32$ means 16 is what part of 32, so $1/3 \div 2/3$ means 1/3 is what part of 2/3. Figure 65, page 74, shows that $1/3 \div 2/3 = 1/2$, since 1/3 is 1/2 of 2/3.

Other problems such as 3/5 ÷ 4/5, 1/4 ÷ 3/4, and 3/8 ÷ 5/8 can be solved by using pictures similar to the one shown in Figure 65. After the solutions have been discovered by using cardboards and rubber bands,




1/3 is 1/2 of 2/3

Figure 65

pictures, etc; the solutions can be summarized as follows:

1/3 ÷ 2/3 = 1/2

3/5 ÷ 4/5 = 3/4

1/4 ÷ 3/4 = 1/3

3/8 ÷ 5/8 = 3/5

From this summarizing list, the student should see that when like fractions are divided, the answer can be obtained by dividing the numerator of the dividend by the numerator of the divisor as is shown below:

1/3 ÷ 2/3 = 1 ÷ 2 = 1/2

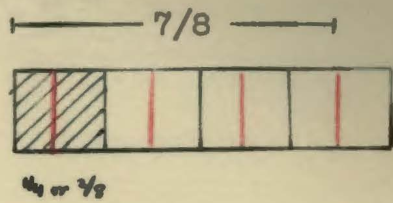
3/5 ÷ 4/5 = 3 ÷ 4 = 3/4

1/4 ÷ 3/4 = 1 ÷ 3 = 1/3

3/8 ÷ 5/8 = 3 ÷ 5 = 3/5

This rule for dividing like fractions by dividing the numerator of the dividend by the numerator of the divisor, has already been used when the divisor of the like fractions is smaller than the dividend. For instance, 3/4 ÷ 1/4 = 3 ÷ 1 = 3.

Thus, unlike fractions with divisors larger than the dividends may be solved by first changing the unlike fractions to like fractions and then by dividing the numerator of the dividend by the numerator of the divisor. As shown in Figure 66, 1/4 ÷ 7/8 = 2/8 ÷ 7/8 = 2 ÷ 7 = 2/7. The exercise, 1/4 ÷ 7/8 means that 1/4 is what part of 7/8? Since 1/4 = 2/8, the exercise is asking what part 2/8 is of 7/8. The shaded part of the diagram is 2 of the 7 equal parts of 7/8. From Figure 66, it can be seen that 1/4 ÷ 7/8 does not equal 1 ÷ 7 . The 1/4 needs to be changed to 8ths so that 2/8 ÷ 7/8 = 2 ÷ 7 = 2/7.



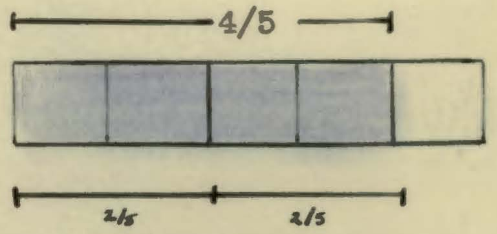

1/4 is 2/7 of 7/8

Figure 66

The "inversion method" of dividing fractions by inverting the divisor and multiplying should be introduced only after the students have a thorough understanding of the division process. Then when the students understand the division process by using concrete and semi-concrete material, as has been described earlier in this chapter, the "inversion method" may be introduced as a short cut when dividing fractions. As explained on page 70, exercises involving whole numbers divided by a fraction may be used when introducing the "inversion method." For example, a diagram similar to Figure 63, page 70, may be used to solve 3 ÷ 2/3 = $4\frac{1}{2}$ .

Then the teacher can show that $1 \div 2/3 = 1\frac{1}{2}$. Three is $3 \times 1$. Therefore, $3 \div 2/3 = 3 \times (1 \div 2/3) = 3 \times 1\frac{1}{2} = 3 \times \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$. By inspection, the students can be shown that 2/3 inverted is 3/2. Thus $3 \div 2/3$ can be solved by inverting 2/3 and multiplying $3 \times 3/2$.

A number of exercises need to be solved by using several different solutions. For instance, $4/5 \div 2/5$ can be solved by using a diagram as shown in Figure 67. Also by




$4/5 \div 2/5 = 2$

Figure 67

using the "common denominator method" $4/5 \div 2/5$ can be solved. $(4/5 \div 2/5 = 4/2 = 2)$ Then $4/5 \div 2/5$ can be solved by using the "inversion method" as follows: $4/5 \div 2/5 = \frac{4}{5} \times \frac{5}{2} = \frac{20}{10} = 2$. All three solutions should be compared and it will be seen that the same answer is obtained by using any of the three methods. By using diagrams and other semi-concrete materials along with the "inversion method" the student should see that the "inversion method" is merely a quick method for solving problems involving the division of fractions.

However, the teacher needs to be sure that the meaning of the division process is understood by the student before the "inversion method" is introduced, because the "inversion method" is mechanical in nature and is not easily understood by the pupil. The pupil needs concrete and semi-concrete activities to lay a foundation of meaning so that the division of fractions can be understood.

The examples used in Chapter IV are illustrative of similar activities which teachers can use to make the division of fractions meaningful to pupils.

## CONCLUSION

As the author has expressed throughout this paper, fractions can be made meaningful to sixth grade students by using concrete and semi-concrete activities. By using concrete and semi-concrete activities similar to those described in the preceding chapters, the students can discover for themselves concepts used in the fundamental processes. When a concept is discovered by a student who is using concrete activities, that concept will probably be understood by that student.

The activities described in this paper for teaching the fundamental processes with fractions are merely representative of many activities that can be used in a classroom by a resourceful teacher. No major effort was made to present the material in a sequential order. It is hoped that teachers who read this paper will adapt the ideas presented to fit their own teaching situation. The activities described can be adapted to supplement any basic textbook in arithmetic.

BIBLIOGRAPHY

Breed, Fredrick S., James R. Overman, and Clifford Woody, Child-Life Arithmetics, Grade Six. Chicago: Lyons and Carnahan, 1937. 293 pp.

Brueckner, Leo J., Foster E. Grossnickle, and Elda L. Merton, Arithmetic We Use, Grade 6. Chicago: The John C. Winston Company, 1942. 312 pp.

Burrows, Alvina Treut, Teaching Children in the Middle Grades. Chicago: D. C. Heath and Company, 1952. 280 pp.

Carpenter, Dale, and Dorothy L. Pepper, The World of Numbers, Arithmetic 6. New York: The Macmillan Company, 1950. 316 pp.

Clark, John R., Monica M. Hoye, and Caroline Hatton Clark, Arithmetic for Young America, Grade 6. New York: World Book Company, Publishers, 1949. 297 pp.

Cronback, Lee J., Educational Psychology. Danville, Illinois: Interstate Printers and Publishers, Inc., 1952. 320 pp.

Dale, Edgar, Audio-Visual Methods in Teaching. New York: The Dryden Press, 1954. 534 pp.

Dawson, Dan T., "Number Grouping as a Function of Complexity," The Elementary School Journal, LIV (September, 1953), 35-42.

Kearney, Noland C., Elementary School Objectives. New York: Russell Sage Foundation, 1953. 180 pp.

Klapper, Paul, The Teaching of Arithmetic. New York: D. Appleton-Century Company, 1934. 525 pp.

Knight, F. B., J. W. Studebacker, and G. M. Ruch, Study Arithmetics, Book 6. Chicago: Scott, Foresman and Company, 1943. 448 pp.

Lansdown, Brenda, "There's More to Mathematics Than the Reckoning," Childhood Education, (February, 1954), 272-279.

Morton, Robert L., Merle Gray, Elizabeth Springston, and William L. Schaaf, Making Sure of Arithmetic, Teacher's Guide, Grade Six. Chicago: Silver Burdett Company, 1947. 226 pp.

_____, Making Sure of Arithmetic, Teacher's Guide, Grade Six. Chicago: Silver Burdett Company, 1955. 225 pp.

Morton, Robert L., Teaching Arithmetic in the Intermediate Grades. Chicago: Silver Burdett Company, 1927. 358 pp.

National Council of Teachers of Mathematics, The Committee on Multi-Sensory Aids, Multi-Sensory Aids in the Teaching of Mathematics. New York: Bureau of Publications, Teacher's College, Columbia University, 1945. 455 pp.

National Council of Teachers of Mathematics, The Tenth Yearbook, The Teaching of Arithmetic. New York: Bureau of Publications, Teacher's College, Columbia University, 1935. 303 pp.

Nickell, Vernon L., Curriculum and Course of Study Guide for Elementary Schools of Illinois. Springfield, Illinois.

Purdy, C. Richard, and Lucien B. Kinney, "Directing Learning in Arithmetic," The Elementary School Journal, LIV (January, 1954), 285-290.

Robinson, Helen M., "Educational News and Editorial Comment," The Elementary School Journal, LIV (May, 1954), 487-488.

Seyfert, Warren C., "What are Fundamentals," The Elementary School Journal, LIV (February 1954), 324-332.

Spitzer, Herbert F., The Teaching of Arithmetic. Chicago: Houghton Mifflin Company, 1954. 397 pp.

Stokes, C. Newton, Teaching the Meaning of Arithmetic. New York: Appleton-Century-Crofts, Inc., 1951. 531 pp.

Strayer, George Drayton, George Willard Frasier, and Winfield Dockery Armentrout, Principles of Teaching. Chicago: American Book Company, 1936. 295 pp.

Wieland, John A., Curriculum Bulletin Number Three. Springfield, Illinois: Printed by the Authority of the State of Illinois, 1950. 79 pp.